MATH 2850: Take Home 05 (50 points.)

NAME: ______________________________

DUE: Wednesday, February 21st, at the beginning of class.

DIRECTIONS: Show all work.

1. Consider the IVP: $y' = \sin(xy)$, $y(0) = 1$.

   (a) Use Picard's EUT to show this IVP has a unique solution on the interval $(-\infty, \infty)$.

   (b) Approximate $y(1)$ using Euler's Method. Use the step sizes $h = 0.5$, $h = 0.25$, and $h = 0.01$.

   Record your approximations to at least five decimal places.

   List your best estimate for $y(1)$ **to as many decimal places you are sure it is accurate**.

   - $h = 0.5$: $y(1) \approx$
   - $h = 0.25$: $y(1) \approx$
   - $h = 0.01$: $y(1) \approx$

   (c) Repeat part (b) using the Improved Euler's and Runge Kutta Fourth Order Methods.

   Improved Euler's Result:

   - $h = 0.5$: $y(1) \approx$
   - $h = 0.25$: $y(1) \approx$
   - $h = 0.01$: $y(1) \approx$

   Runge Kutta Result:

   - $h = 0.5$: $y(1) \approx$
   - $h = 0.25$: $y(1) \approx$
   - $h = 0.01$: $y(1) \approx$

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2. The path $y = f(x)$ taken by a dog chasing a stick which is thrown from $(0, 0)$ straight up the positive y-axis is modeled by the second order IVP:

$$x \frac{d^2 y}{d x^2} = k \sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \qquad y(p) = 0, \quad y'(p) = 0.$$

Here we assume $x > 0$, $k > 0$ is the ratio of the stick's speed to the dog's speed (both of which we assume are constant), and $p > 0$ is the starting position of the dog on the x-axis.[1]

(a) Let $v = \dfrac{dy}{dx}$. Show the IVP for y gives the following IVP for v: $x \dfrac{dv}{dx} = k \sqrt{1 + v^2}$, $v(p) = 0$.

Show the resulting DE for v is separable and put it in the form: $h(v)\, dv = g(x)\, dx$.

NOTE: Do **NOT** integrate the DE just yet!

[1]Deriving this IVP is a wonderful project which combines physics, vector calculus, and differential equations!

(b) **INTERLUDE:** How do find $\displaystyle\int \frac{1}{\sqrt{1+v^2}}\,dv$?

In a previous Calculus course, you probably learned to exploit the (circular) trigonometric identity:

$$1+\tan^2(\theta)=\sec^2(\theta),$$

by making the substitution $v=\tan(\theta)$ so that $dv=\sec^2(\theta)\,d\theta$. The integral becomes:

$$\int \frac{1}{\sqrt{1+v^2}}\,dv=\int \frac{\sec^2(\theta)}{\sqrt{1+\tan^2(\theta)}}\,d\theta=\ldots=\int \sec(\theta)\,d\theta=\ldots=\ln\left(v+\sqrt{1+v^2}\right)+C$$

Recall the hyperbolic functions also have Pythagorean-like identities we can exploit!

In this case, we have the identity: $1+\sinh^2(t)=\cosh^2(t)$.

Evaluate $\displaystyle\int \frac{1}{\sqrt{1+v^2}}\,dv$ using the substitution: $v=\sinh(t)$ to show: $\displaystyle\int \frac{1}{\sqrt{1+v^2}}\,dv=\sinh^{-1}(v)+C$

NOTE: $\sinh^{-1}(v)$ denotes the **inverse** hyperbolic sine function, so in particular, $\sinh^{-1}(v)\neq \dfrac{1}{\sinh(v)}$.

HELPFUL FACTS TO RECALL: $D_t[\sinh(t)]=\cosh(t)$ and $\cosh(t)>0$ for all t.

(c) Use your work in part 2b to help you solve IVP from 2a for $\sinh^{-1}(v)$.

Don't forget to impose the initial condition.

(d) Show your solution to part 2c can be written as: $\sinh^{-1}(v) = \ln\left(\frac{x^k}{p^k}\right)$.

(e) Starting with the answer in part 2d, use the definition of $\sinh(t) = \frac{1}{2}\left(e^{t} - e^{-t}\right)$ to show

$$v = \frac{1}{2}\left(\frac{x^k}{p^k} - \frac{p^k}{x^k}\right)$$

(f) Substitute $v = \dfrac{dy}{dx}$ into your solution for part 2e and integrate to solve for the path $y = f(x)$.

NOTE: There are **two** cases, depending on if $k = 1$ or not (do you see why?)

Don't forget to impose the initial condition, $y(p) = 0$ for each case.

- Solution assuming $k = 1$.

- Solution assuming $k \neq 1$.

(g) Let's assume $p = 1$. Use desmos to graph the solution curves for $k = 0.5$, $k = 1$, and $k = 1.5$.

- $k = 0.5$ case: pursuit curve $y = f(x)$:

  Does the dog catch the stick? How do you know by looking at the graph?

- $k = 1$ case: pursuit curve $y = f(x)$:

  Does the dog catch the stick? How do you know by looking at the graph?

- $k = 1.5$ case: pursuit curve $y = f(x)$:

  Does the dog catch the stick? How do you know by looking at the graph?

(h) Use your solution to part 2f to algebraically show that if $k = 1$, the dog never catches the stick. What is the relationship between the speed of the stick and the speed of the dog in this scenario?

(i) Use your solution to part 2f algebraically determine under which circumstances the dog catches the stick if $k \neq 1$. What is the relationship between the speed of the stick and the speed of the dog in each of these scenarios?