MATH 1650: TAKE HOME 03 - 20 POINTS

NAME: ____

DIRECTIONS: To receive full credit, make sure your work is neat and complete.

SECTION 5.1, 5.2, and 5.3 PRACTICE PROBLEMS

1. Let $f(x) = \dfrac{\sqrt{9x^2+1}}{5x}$.

   (a) Analytically determine if f is even, odd, or neither.

   (b) Check your answer using desmos.

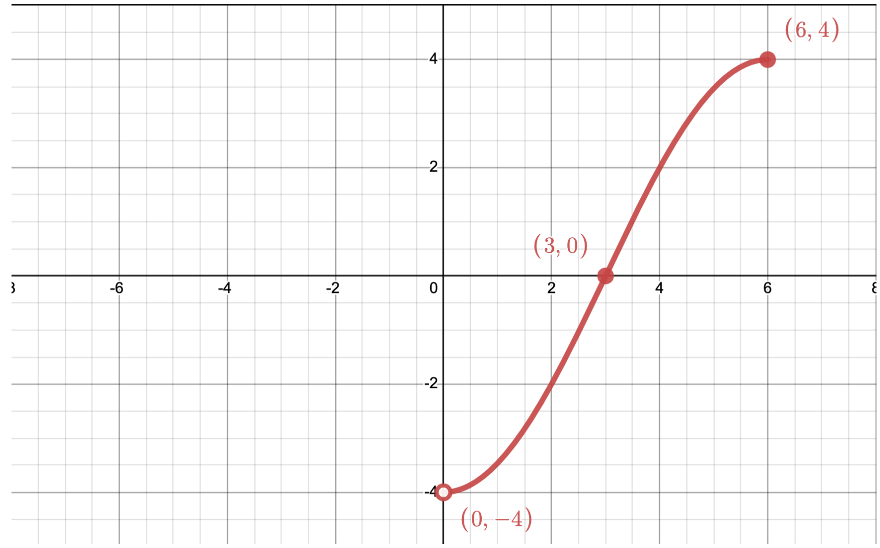
2. Let $f(x) = x^3 - 0.0001x^2 + 3x$

   (a) Graph $y = f(x)$ using desmos. Does f appear to be odd? Explain.

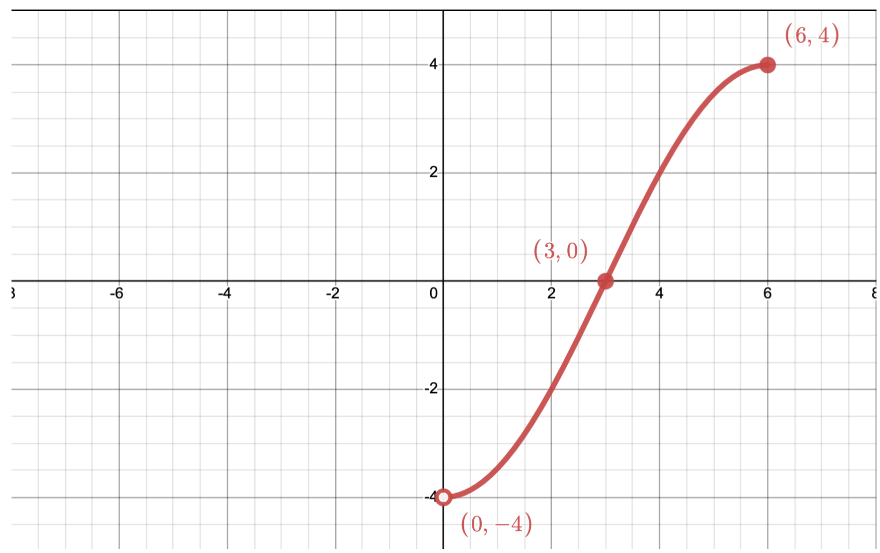
   (b) Find $f(-1)$ and $f(1)$. Is f odd? Explain.

3. Complete the graph of f below assuming:

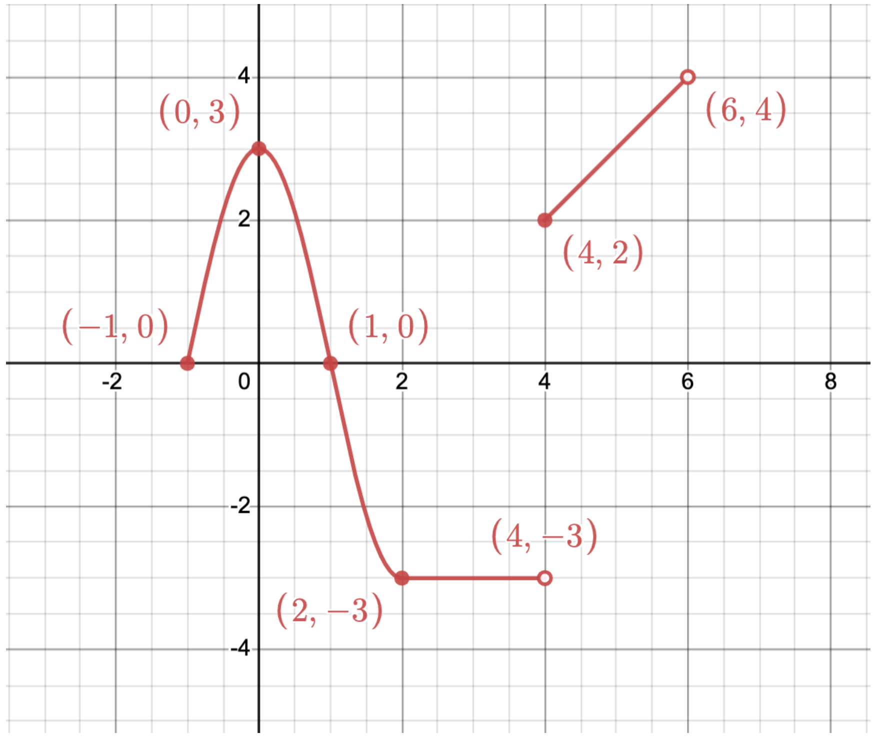
(a) f is even:



(b) f is odd:



4. Answer the following questions based on the complete graph of $y = f(x)$ below:



(a) List the domain and range of f using interval notation.

- domain:
- range:

(b) Find the maximum of f, if it exists:

Find the minimum of f, if it exists:

(c) List the zeros of f:

(d) List the intervals over which f is increasing, decreasing, and constant.

- increasing:
- decreasing:
- constant:

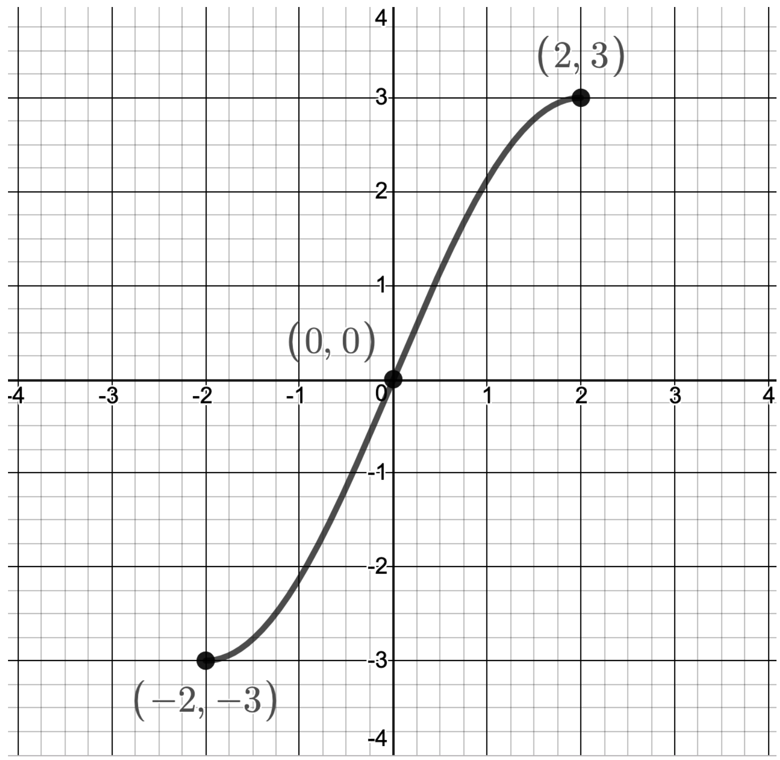
(e) Explain why $f(4) \neq -3$. What is $f(4)$?

(f) Explain why $(2, -3)$ is a local minimum.

(g) Explain why $(3, -3)$ is both a local minimum and local maximum.

(h) Find $(f \circ f)(-1)$.

5. Consider the graph of $y = f(x)$ below and let $g(t) = t + 2$.



$y = f(x)$

Find and simplify:

(a) $f(2)$

(b) $g(2)$

(c) $(g - f)(2)$

(d) $(fg)(2)$

(e) $(f \circ g)(0)$

(f) $(g \circ f)(2)$

(g) Find the domain of f and g. Write your answers in interval notation:

- domain of f:
- domain of g:

(h) What is the domain of $\dfrac{f}{g}$? Explain your answer.

6. Find functions f and g which decompose the given function h as required.

(a) $h(x) = x^2 + 6x$. Find functions f and g so that $h = f + g$.

(b) $h(x) = x - x^{3/2}$. Find functions f and g so that $h = f - g$.

(c) $h(x) = x\sqrt[3]{x+3}$. Find functions f and g so that $h = fg$.

(d) $h(x) = \dfrac{x^2 - 1}{x + 2}$. Find functions f and g so that $h = \dfrac{f}{g}$.

(e) $h(x) = \sqrt{x^2 + 4}$. Find functions f and g so that $h = f \circ g$.

7. For $f(x) = x^2 - 3x$, find and simplify:

- $f(x+h)$
- $\dfrac{f(x+h) - f(x)}{h}$

$f(x+h) =$

$$\frac{f(x+h) - f(x)}{h} =$$

8. For the following pairs of functions, find and simplify an expression for $(f \circ g)(x)$ and $(g \circ f)(x)$

(a) $f(x) = \dfrac{3x+2}{x+1}$ and $g(x) = \dfrac{2-x}{x-3}$

$(f \circ g)(x) =$ ______________________ $(g \circ f)(x) =$ ______________________

(b) $f(x) = \sqrt{3x-2}$ and $g(x) = \dfrac{x^2+2}{3}$.

$(f \circ g)(x) =$ ____________________ $(g \circ f)(x) =$ ____________________

SECTION 5.6 PRACTICE PROBLEMS

1. Let $f(x) = \dfrac{3x-2}{2x+1}$.

   (a) How can you tell algebraically the graph of f will have a vertical asymptote?

   What is the vertical asymptote?

   (b) How can you tell algebraically the graph of f will have a horizontal asymptote?

   What is the horizontal asymptote?

   (c) Find the axis intercepts of the graph of f:

   - x-intercept:

   - y-intercept:

(d) Graph f using desmos and check your answers to parts (a) through (c).

How does the graph of f suggest f is invertible?

(e) Based on your answers to parts (a) through (c), answer the following questions:

- What is the vertical asymptote of the graph of f^{-1}?

- What is the horizontal asymptote of the graph of f^{-1}?

- What is the x-intercept of the graph of f^{-1}?

- What is the y-intercept of the graph of f^{-1}?

(f) Find a formula for $f^{-1}(x)$.

(g) Check your answer to part (f) by simplifying $\left(f^{-1} \circ f\right)(x)$ and $\left(f \circ f^{-1}\right)(x)$

2. The price $p(x)$ of a game system, in dollars per system, as a function of systems sold weekly, x is given by:

$$p(x) = 450 - 15x \qquad 0 \leq x \leq 30.$$

(a) Find and interpret $p(23)$.

(b) Find $p^{-1}(x)$ and state its domain.

- $p^{-1}(x) =$
- domain:

(c) Find and interpret $p^{-1}(105)$.

(d) The profit (in dollars) made from producing and selling x systems per week is given by:

$$P(x) = -15x^2 + 350x - 2000, \qquad 0 \le x \le 30.$$

Find and simplify $\left(P \circ p^{-1}\right)(x)$ and state its domain.

- $\left(P \circ p^{-1}\right)(x) =$
- domain:

(e) Find and interpret $\left(P \circ p^{-1}\right)(105)$.

(f) Use the vertex formula to determine what price per system should be charged to maximize profit. How many systems will be sold at that price? What is the maximum profit?

NOTE: It's only possible to sell a whole number of systems, so you will have to take your answers and make adjustments ...

3. Let $f(x) = x^2 - 6x + 5$ for $x \leq 3$.

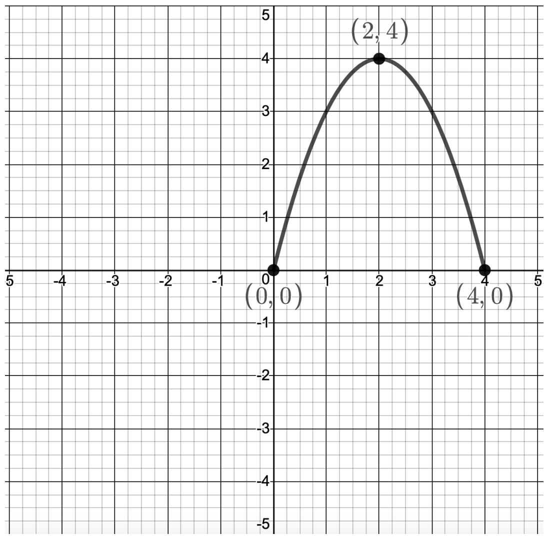
(a) Explain why f is one-to-one.

(b) Find a formula for $f^{-1}(x)$.

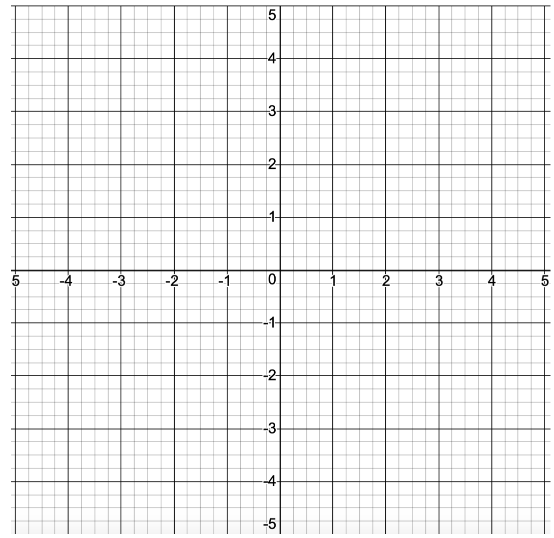
(c) Verify: $f^{-1}(f(x)) = x$ and $f\left(f^{-1}(x)\right) = x$.

SECTION 5.4 PRACTICE PROBLEMS

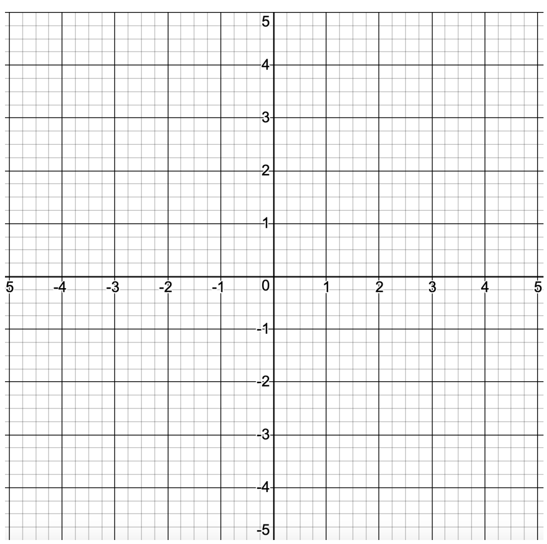
1. Use the graph of f to graph each of the following functions. Label at least three points on each graph.



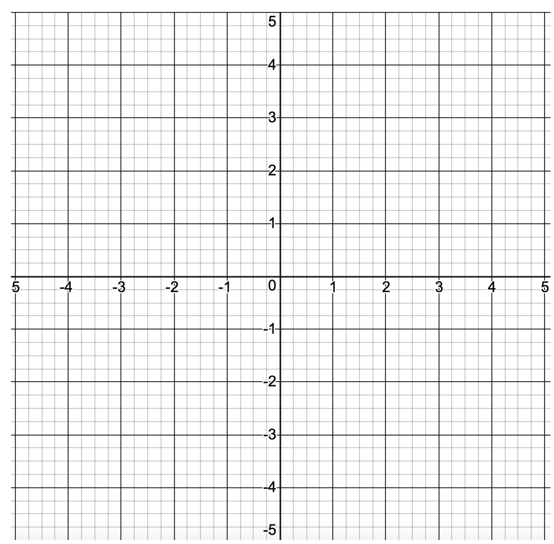
$y = f(x)$



$y = \frac{1}{2}f(x - 1)$

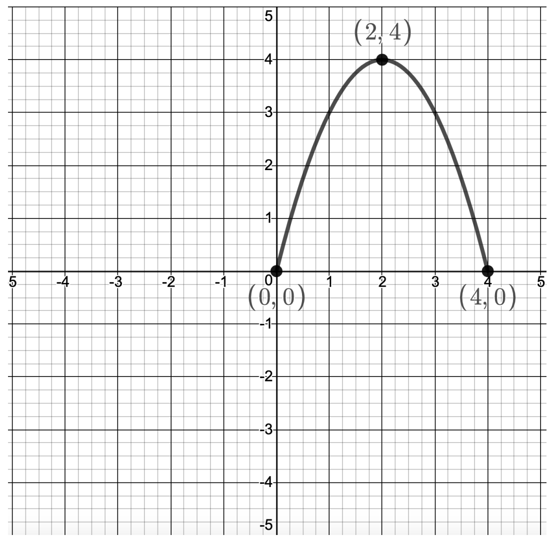


$y = f(-x) - 3$

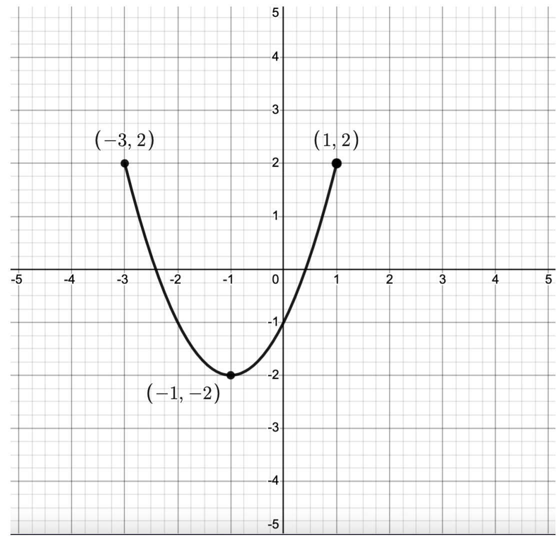


$y = f(2x + 4)$

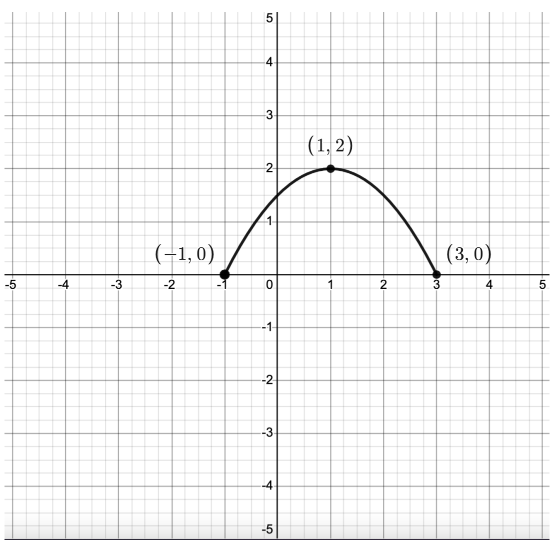
2. Use graph of f below to write a formula for the graphs of the other functions as transformations of f.



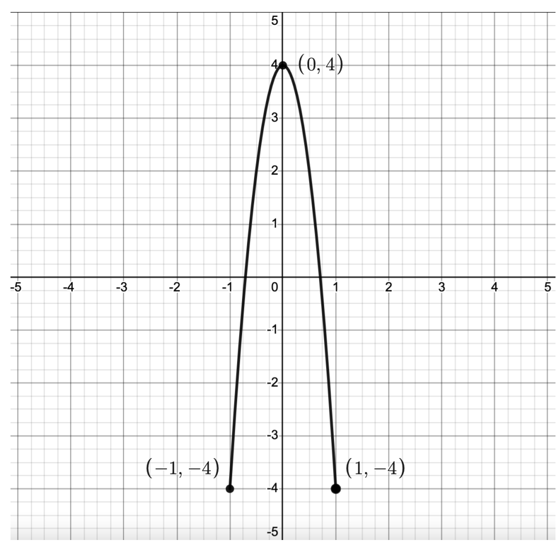
$y = f(x)$



$y =$ ____________



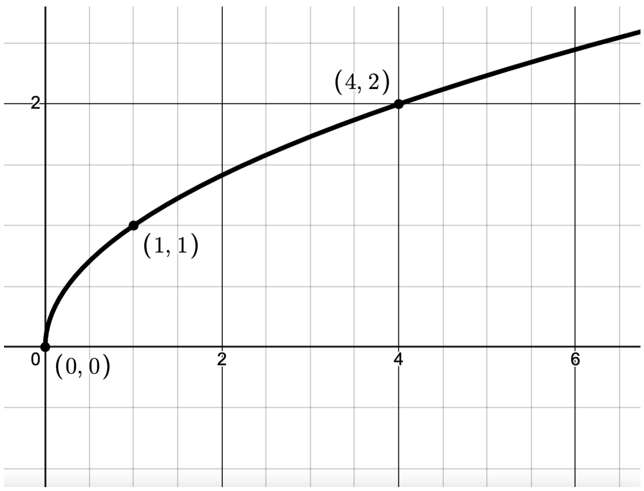
$y =$ ____________

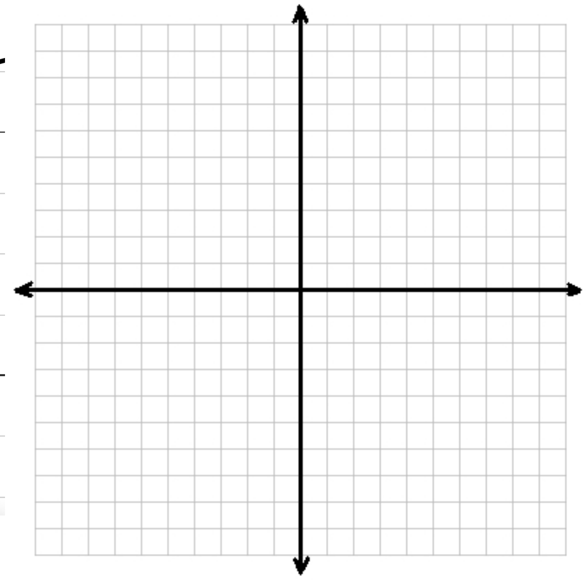


$y =$ ____________

3. For each pair of functions f and g below, start with the graph of $y = f(x)$ and use transformations to graph $y = g(x)$. Track the indicated points / asymptotes through the transformation.

(a)

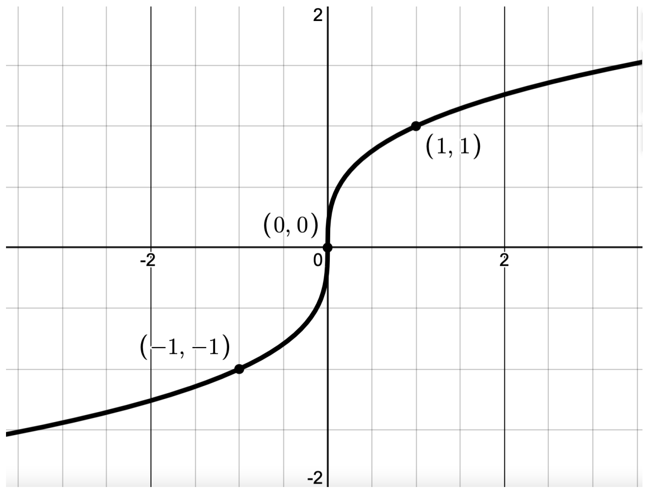


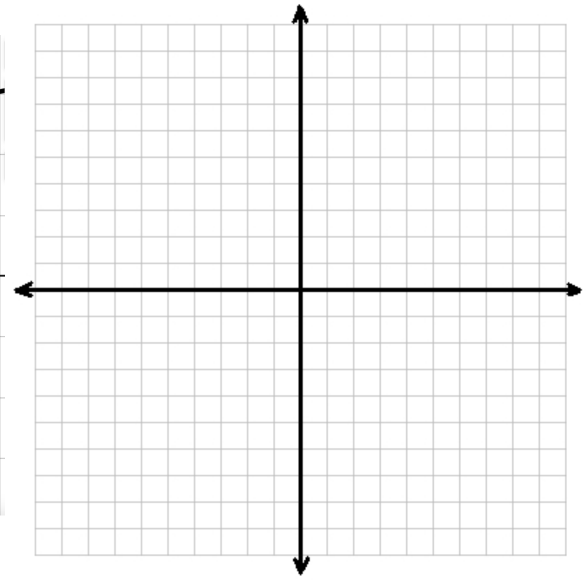


$y = f(x) = \sqrt{x}$

$y = g(x) = 1 - \sqrt{x+2}$

(b)

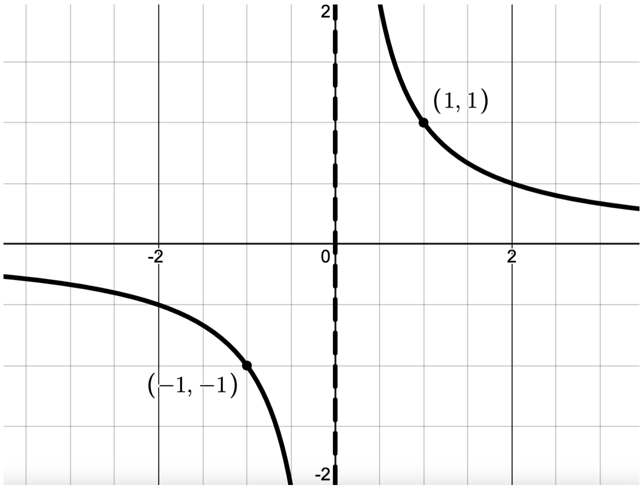


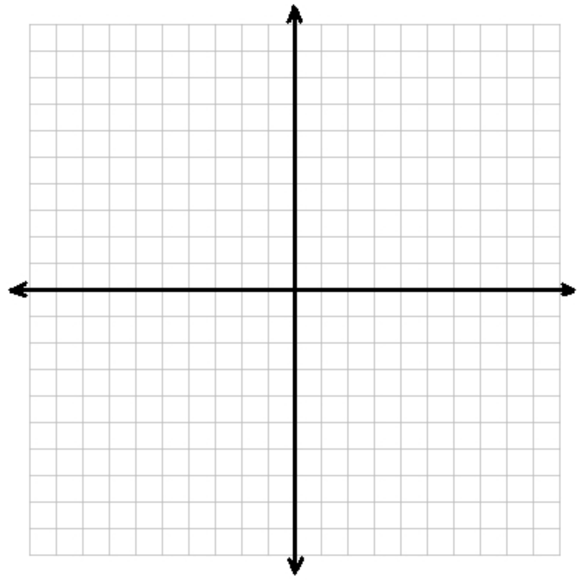


$y = f(x) = \sqrt[3]{x}$

$y = g(x) = \sqrt[3]{2x - 3}$

(c)





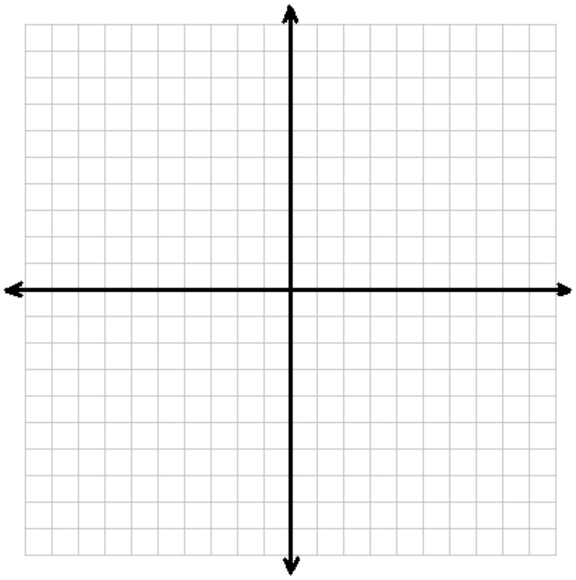
$$y = f(x) = \frac{1}{x}$$

$$y = g(x) = \frac{2x-3}{x+1}$$

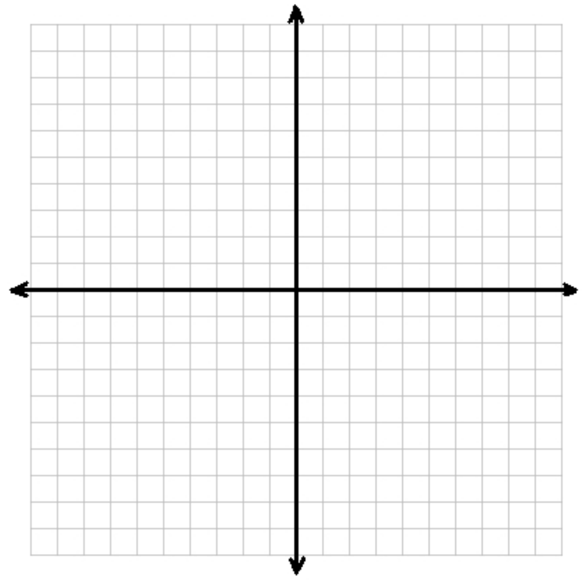
HINT: Use division to rewrite $g(x) = \dfrac{2x-3}{x+1}$ ...

4. **EXPLORATION:** In parts (a) - (c) below, graph of $y = f(x)$ on the left and $y = |f(x)|$ on the right.

(a) $f(x) = x^2 - 4$

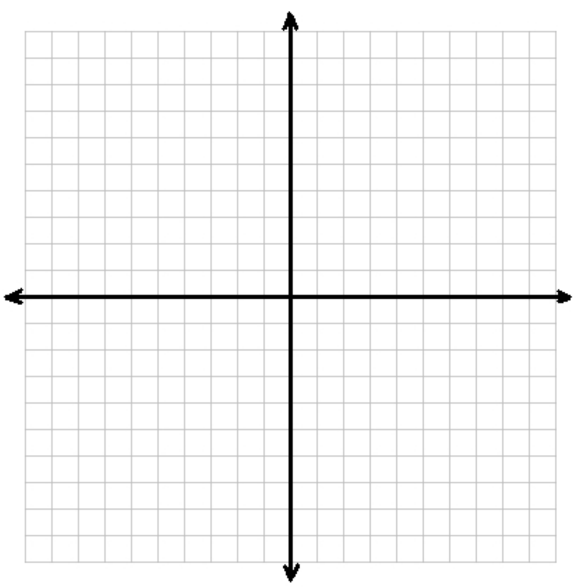


$y = f(x) = x^2 - 4$

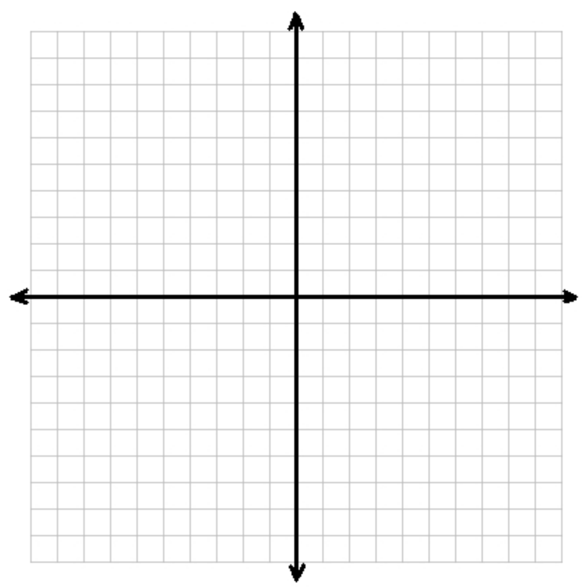


$y = |f(x)| = \left|x^2 - 4\right|$

(b) $f(x) = \dfrac{1}{x}$

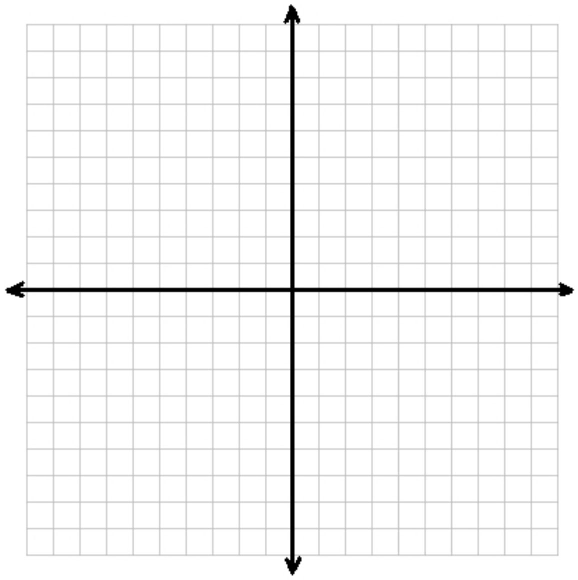


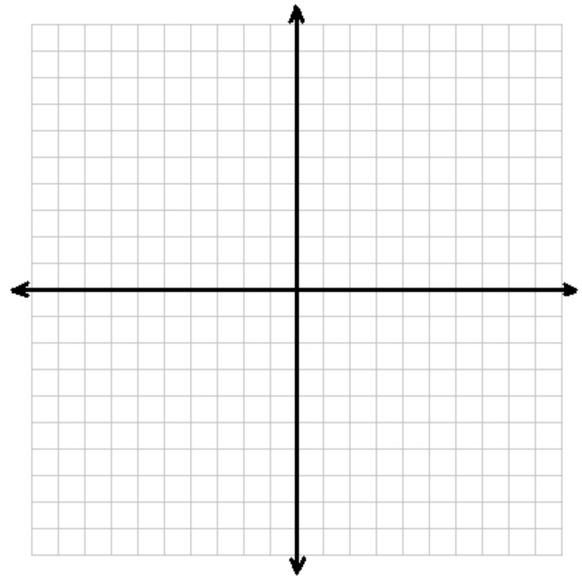
$y = f(x) = \dfrac{1}{x}$



$y = |f(x)| = \left|\dfrac{1}{x}\right|$

(c) $f(x) = \sqrt[3]{x-2}$



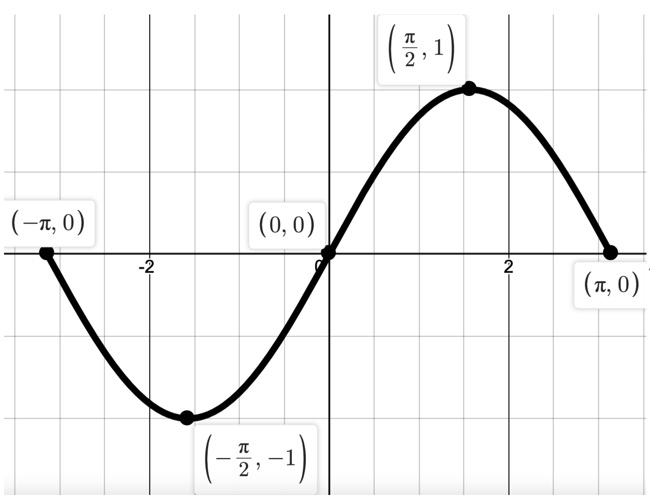


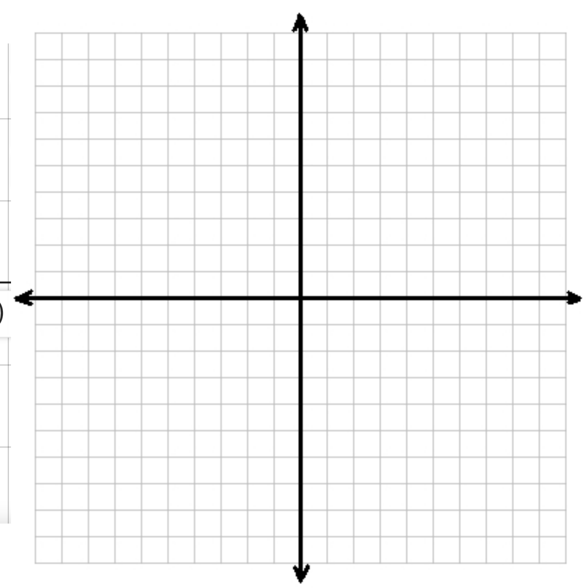
$y = f(x) = \sqrt[3]{x-2}$ $\qquad$ $y = |f(x)| = \left|\sqrt[3]{x-2}\right|$

What observations can you make? How does what is happening graphically connect with the piecewise definition of the absolute value function discussed in Section 1.3?

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

(d) Given the graph below on the left of $y = f(x)$, graph $y = |f(x)|$ below on the right.



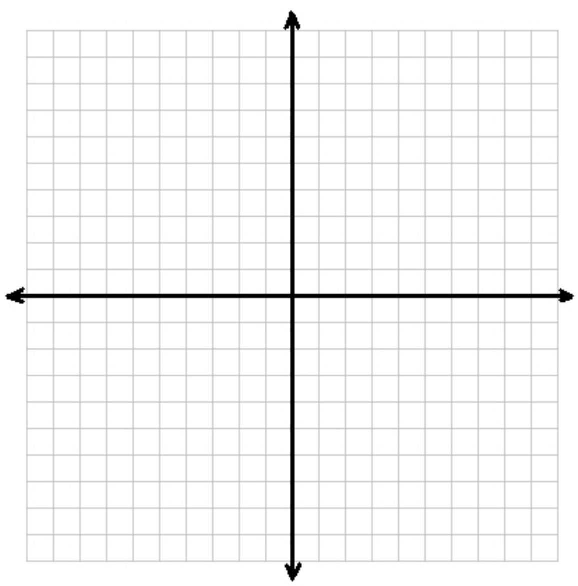


$y = f(x)$ $\qquad$ $y = |f(x)|$

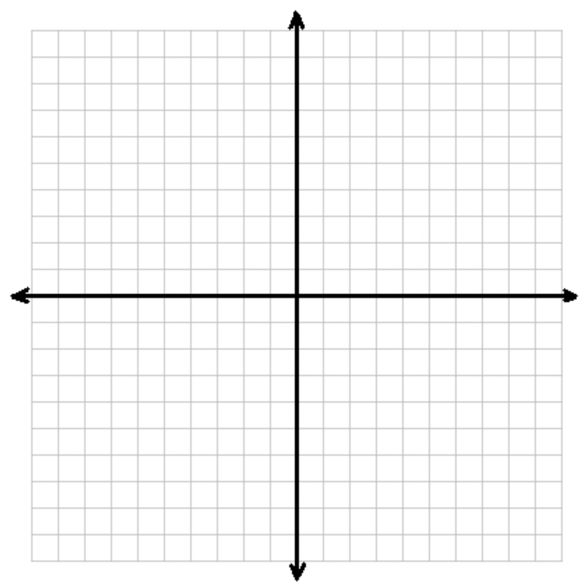
SECTION 6.1 and 6.2 PRACTICE PROBLEMS

1. Let $f(x) = 2^{x+3} - 1$

   (a) Starting with the graph of $y = 2^x$, graph $y = f(x)$ using transformations.
   Track the points $(-1, 0.5)$, $(0, 1)$, $(1, 2)$ and the horizontal asymptote through the transformations.



$y = 2^x$



$y = f(x)$

   (b) From your graph, determine the domain and range of f.

   domain: ______________________________ range: ______________________________

(c) Explain how the graph of f suggests f is one-to-one.

(d) Find a formula for $f^{-1}(x)$ using the 'procedural' method shown in class.

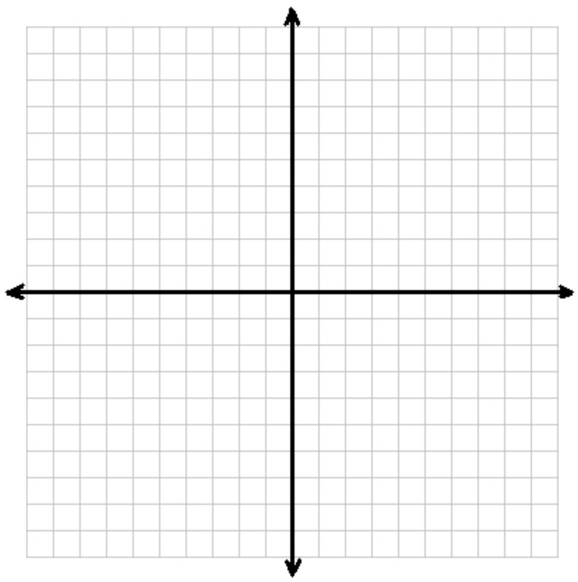
(e) Check your answer to part (c) by simplifying $\left(f^{-1} \circ f\right)(x)$ and $\left(f \circ f^{-1}\right)(x)$:

- $\left(f^{-1} \circ f\right)(x) =$

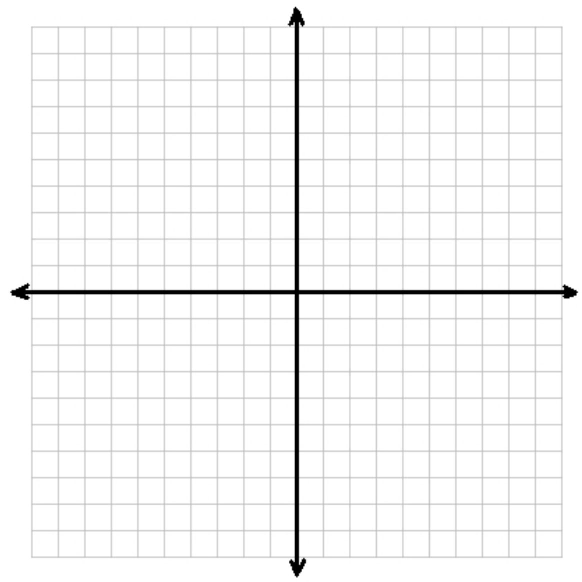
- $\left(f \circ f^{-1}\right)(x) =$

(f) Use the graph of f to graph f^{-1}.

Label at least three corresponding points and the asymptotes on each graph.



$y = f(x)$



$y = f^{-1}(x)$

(g) From your graph, determine the domain and range of f^{-1}.

domain: ____________________________ range: ____________________________

2. Write a sentence (or two!) to explain why $\ln(-1)$ is not a real number.

3. Write a sentence (or two!) to explain why $\log(0)$ is not a real number.

4. Let $f(x) = \dfrac{5-x}{x+2}$.

(a) Make a Sign Diagram for f.

(b) Use your Sign Diagram in part (a) to determine the domains of the following functions:

i. $g(x) = \sqrt{\dfrac{5-x}{x+2}}$

ii. $h(x) = \sqrt[3]{\dfrac{5-x}{x+2}}$

iii. $e(x) = e^{\frac{5-x}{x+2}}$

iv. $l(x) = \ln\left(\dfrac{5-x}{x+2}\right)$

5. Let $f(x) = e^{2x} - 1$ and $g(x) = \frac{1}{2}\ln(x+1)$.

(a) Find and simplify a formula for $(f \circ g)(x)$. Write out each and every step of the simplification.

(b) Find and simplify a formula for $(g \circ f)(x)$. Write out each and every step of the simplification.

(c) Based on your answers to (a) and (b), are f and g inverses? Explain.

6. **EXPLORATION:** Use desmos to graph each pair of functions f and g. What do you notice?

(a) i. $f(x) = e^x$ and $g(x) = 1 + x + \dfrac{x^2}{2}$

ii. $f(x) = e^x$ and $g(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24}$

iii. $f(x) = e^x$ and $g(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + \dfrac{x^5}{120} + \dfrac{x^6}{720}$

iv. Do you see the pattern?

(b) i. $f(x) = \ln(x)$ and $g(x) = (x-1)$

ii. $f(x) = \ln(x)$ and $g(x) = (x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3}$

iii. $f(x) = \ln(x)$ and $g(x) = (x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3} - \dfrac{(x-1)^4}{4} + \dfrac{(x-1)^5}{5}$

iv. Do you see the pattern?

MATH 1650: SECTION 6.3 PRACTICE PROBLEMS

1. Expand the given logarithm and simplify. Assume all quantities represent positive real numbers.

   (a) $\log_2\left(\dfrac{128}{x^2+4}\right)$

   (b) $\log(1.23 \times 10^{37})$

   (c) $\log_5\left(x^2 - 25\right)$

   (d) $\log_{\frac{1}{3}}(9x(y^3 - 8))$

   (e) $\log_3\left(\dfrac{x^2}{81y^4}\right)$

   (f) $\log_6\left(\dfrac{216}{x^3y}\right)^4$

   (g) $\log_{\frac{1}{2}}\left(\dfrac{4\sqrt[3]{x^2}}{y\sqrt{z}}\right)$

2. Use the properties of logarithms to write the expression as a single logarithm.

(a) $\log_2(x) + \log_2(y) - \log_2(z)$

(b) $\frac{1}{2}\log_3(x) - 2\log_3(y) - \log_3(z)$

(c) $\log(x) - \frac{1}{3}\log(z) + \frac{1}{2}\log(y)$

(d) $\log_5(x) - 3$

(e) $\log_7(x) + \log_7(x-3) - 2$

(f) $\log_2(x) + \log_4(x)$

(g) $\log_2(x) + \log_{\frac{1}{2}}(x-1)$

3. Convert the following functions to functions base 'e'. Check your answers graphically using desmos.

(a) $f(x) = 9^x$

(b) $g(x) = \log_9(x)$

4. Suppose HA represents a weak acid. Then we have a reversible chemical reaction

$$HA \rightleftharpoons H^+ + A^-.$$

The acid disassociation constant, K_a, is given by

$$K_a = \frac{[H^+][A^-]}{[HA]} = [H^+]\frac{[A^-]}{[HA]},$$

where the square brackets denote the concentrations.

The symbols pH and pK_a are defined as $\text{pH} = -\log([H^+])$ and $\text{p}K_a = -\log(K_a)$. Using Properties of logarithms, derive the Henderson-Hasselbalch Equation:

$$\text{pH} = \text{p}K_a + \log\frac{[A^-]}{[HA]}$$

SECTION 6.4 and 6.5 PRACTICE PROBLEMS

1. Solve each equation analytically. Check your answers graphically.

   (a) $3^{(x-1)} = 27$

   (b) $9 \cdot 3^{7x} = \left(\frac{1}{9}\right)^{2x}$

   (c) $e^{-5730k} = \frac{1}{2}$

   (d) $30 - 6e^{-0.1t} = 20$

(e) $\dfrac{5000}{1+2e^{-3t}} = 2500$

(f) $7e^{2t} = 28e^{-6t}$

(g) $e^{2t} - 3e^{t} - 10 = 0$

(h) $e^{x} + 15e^{-x} = 8$

2. Solve each equation analytically. Check your answers graphically.

(a) $\log_5\left(18 - t^2\right) = \log_5(6 - t)$

(b) $\log_{\frac{1}{2}}(2x - 1) = -3$

(c) $\log\left(\frac{x}{10^{-3}}\right) = 4.7$

(d) $3\ln(t) - 2 = 1 - \ln(t)$

(e) $\log_5(2t + 1) + \log_5(t + 2) = 1$

(f) $\log_{16}(x-2)+\log_{16}(x+1)=\dfrac{1}{2}$

(g) $\ln(\ln(x))=3$

(h) $\ln(t^2)=(\ln(t))^2$

3. For each equation below:

   - Explain why the techniques of Sections 6.4 and 6.5 are ineffective at solving the following equations.
   - Use desmos to solve the equation graphically. Round answers to three decimal places.

   (a) $e^{x-1}=3x$

   (b) $\ln(2x-1)=x^2-4x-3$

SECTION 6.6 PRACTICE PROBLEMS

1. Suppose \$1000 is invested in an account which offers 0.5 % interest, compounded continuously.

   (a) Find the amount A in the account as a function of the term of the investment t in years.

   (b) To the nearest cent, determine how much is in the account after 5, 10, 30 and 35 years.

   (c) To the nearest year, determine how long will it take for the initial investment to double.

2. The radioactive element Chromium 51, used to track red blood cells, has a half-life of 27.7 days.

(a) Find the decay constant k. Round your answer to four decimal places.

(b) If 75 milligrams is present initially, find a function which gives the amount of Chromium 51, $A(t)$, which remains after t days.

(c) Determine how long it takes for 90% of theChromium 51 to decay, rounded to two decimal places.

3. Suppose $V(t) = 7500(0.85)^t$ is the value of a car (in dollars) t years after it is purchased.

   (a) Find and interpret $V(0)$ and $V(1)$.

   (b) Find and interpret $\dfrac{V(1) - V(0)}{V(0)}$.

   (c) Determine how long it takes for the car to depreciate to half its purchase price algebraically. Find an exact answer then find an approximation rounded to the nearest year.

4. The population of a certain species of bird in a local sanctuary, $P(t)$, t years after 2020 is given by:

$$P(t) = \frac{2000}{1 + 4e^{-0.1t}}, \quad t \geq 0$$

Solve the equation $P(t) = 1500$ algebraically and interpret your answer in terms of the bird population.