MATH 2700: TEST 05 (100 points.)

NAME: ______________________________

DIRECTIONS: Show all work.

1. Let C be the semicircle $y = \sqrt{9 - x^2}$.

   (a) What is the radius of the semicircle?

   (b) Find a smooth parametrization of C, oriented counter-clockwise.

   (c) Use your answer to (b) to calculate the lateral surface between C and the lift of C to $z = x^2 y$.

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2. Let C be the portion of the helix: $\vec{r}(t) = \langle \sin(t), t, \cos(t) \rangle$, $0 \leq t \leq 2\pi$.

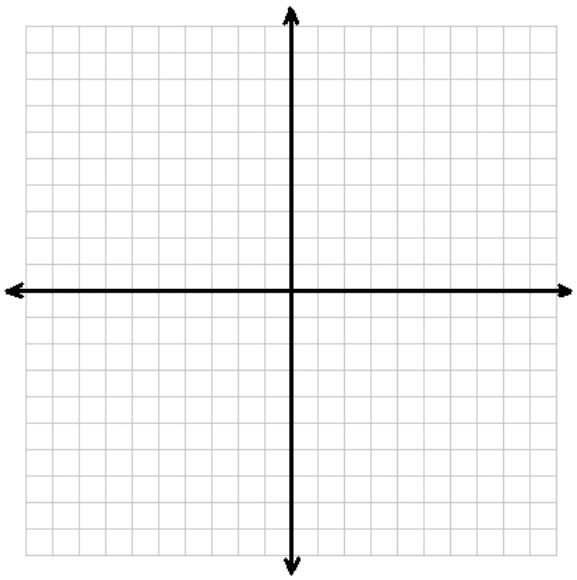
(a) Show the field $\vec{F}(x, y, z) =< z, -4y, x >$ is conservative by finding a potential $\phi(x, y, z)$ for $\vec{F}(x, y, z)$.

(b) Find the work done moving along C through $\vec{F}$ using the Fundamental Theorem of Line Integrals.

3. Let C be the triangle in the first quadrant with vertices $(0,0)$, $(4,0)$ and $(0,2)$ oriented counter-clockwise.

   Let $\vec{F}$ be the field $\vec{F}(x,y) = \langle -y, x \rangle$.

   (a) Sketch C and find a three part parameterization for C. Label each piece on the graph.



   (b) Find the **circulation** of $\vec{F}(x,y)$ **along** C using line integral(s).

(c) Find the **outward flux** of $\vec{F}(x, y)$ **across** C using line integral(s).

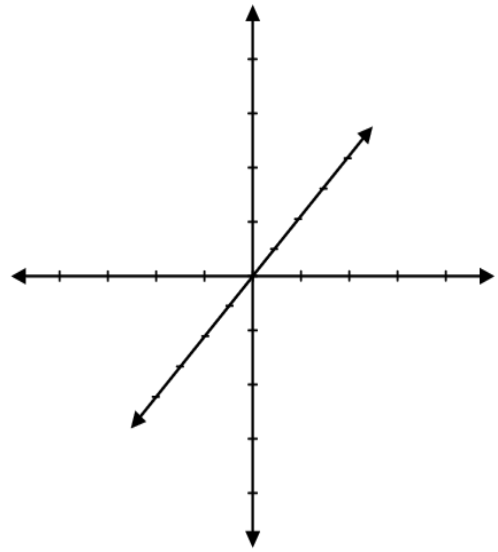
(d) Check your answers to parts (b) and (c) using Green's Theorem.

- Circulation (from part (b)):

- Flux (from part (c)):

4. Let S be given by: $\vec{r}(u, v) = \left\langle u^2, u\cos(v), u\sin(v)\right\rangle$, $0 \leq u \leq 4$, $0 \leq v \leq 2\pi$.

(a) Convert the given representation to rectangular coordinates to sketch or otherwise describe the graph.



(b) Find and simplify an expression for $\vec{r}_u \times \vec{r}_v$. **HINT:** 'Pythagorean Magic.'

(c) Write the equation of the tangent plane to this surface at the point $(9, 0, 3)$.

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(d) Set-up but do not evaluate[1] double iterated integrals with order $du\,dv$ which would calculate:

i. the surface area of S.

ii. the flux of $\vec{F}(x, y, z) = \langle 2y, 1, x \rangle$ across S parallel to $\vec{r}_u \times \vec{r}_v$.

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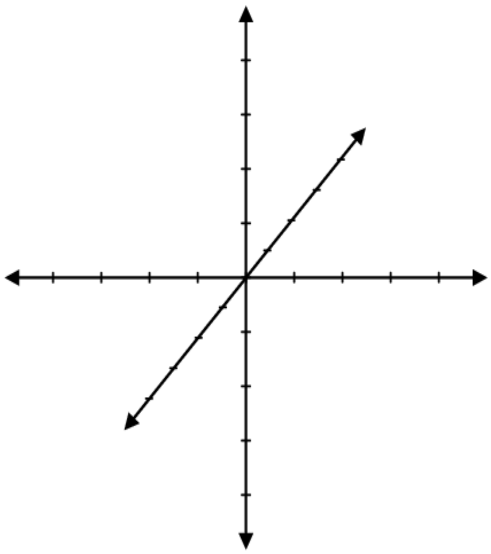
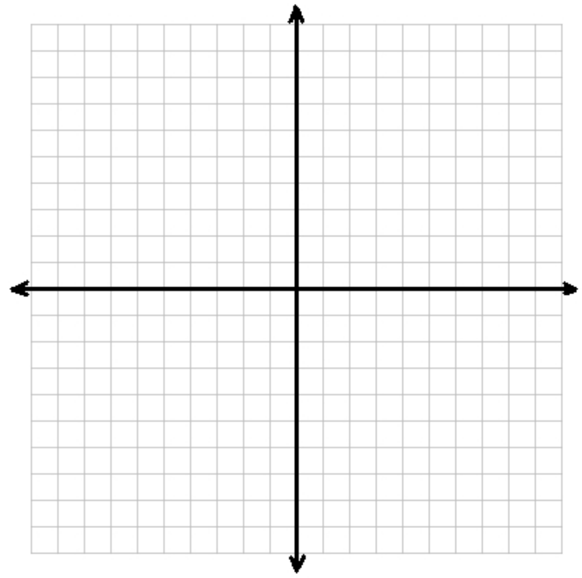
[1]but please simplify the integrands!

5. Let S be the portion of $5x + y + 2z = 10$ which lies in the first octant and let Q be the tetrahedron bounded by S and the coordinate planes. Let C be the boundary of S oriented by the **upward** pointing normal and let $\vec{F}(x, y, z) = \langle 2x, -3z, 2y \rangle$.

(a) Sketch S and its projection into the xy-plane.





(b) Find the divergence and curl of $\vec{F}(x, y, z)$:

i. div $\left(\vec{F}\right)$:

ii. curl $\left(\vec{F}\right)$:

(c) Use Stokes's Theorem to find the work done moving along C by computing an equivalent double integral. Feel free to use properties of double integrals to simplify your computation!

(d) Use the Divergence Theorem to find the outward flux of $\vec{F}$ across the boundary of Q by evaluating an equivalent triple integral. Feel free to use properties of triple integrals to simplify your computation!

BONUS:

1. Prove if R is a simply connected region with boundary C, oriented counter-clockwise, $\oint_C x\,dy =$ the area of R.

2. Use #1 to find the area enclosed by: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

3. If $a = b$, what does your formula from #2 reduce to? What is the shape in this case?

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FORMULAS

$$\int_C \nabla\phi \cdot d\vec{r} = \phi(\text{ending point of } C) - \phi(\text{starting point of } C) \qquad \text{and} \qquad \oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\; dA$$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{N}\,dS = \int_C \vec{F} \cdot d\vec{r} \qquad \text{and} \qquad \iiint_Q \nabla \cdot \vec{F}\,dV = \iint_S \vec{F} \cdot \hat{N}\,dS$$

$$V = \frac{1}{3}\,(\text{area of base})\,(\text{height})$$