MATH 2700: TAKE HOME 05 (50 points.)

NAME: ______________________________

DUE: The day of Test 5, at the beginning of class.

DIRECTIONS: Show all work.

1. Let C be the portion of the helix: $\vec{r}(t) = \langle 3t, 4\cos(t), 4\sin(t) \rangle$, $0 \leq t \leq \pi$.

   Find the work done moving an object along C through $\vec{F}(x, y, z) = < 2x, z, -y >$ using a line integral.

2. Let C be the portion of the helix: $\vec{r}(t) = \langle 3t, 4\cos(t), 4\sin(t) \rangle$, $0 \leq t \leq \pi$.

   NOTE: This is the same curve from $\#1$

   Suppose C models a spring with linear density given by: $\rho(x, y, z) = x\sqrt{y^2 + z^2}$ with units of $\frac{\text{mass}}{\text{length}}$.

   Find and **interpret** $\int_C \rho(x, y, z)\, ds$.

3. Let C be the portion of $x^2 + y^2 = 4$ which lies in the first quadrant.

   Find the lateral surface area between C and the lift of C to the surface $z = x^3 y$ using a line integral.

4. Let C be the triangle with vertices $(0,0)$, $(3,0)$, and $(0,4)$, oriented counter-clockwise.

   Find the (outward) flux of $\vec{F}(x,y) = \left\langle \frac{x}{2}, \frac{y}{2} \right\rangle$ across C by evaluating three line integrals.

   (a) Flux across the side with vertices $(0,0)$ and $(3,0)$:

   (b) Flux across the side with vertices $(3,0)$ and $(0,4)$:

(c) Flux across the side with vertices $(0, 0)$ and $(0, 4)$:

(d) Total flux:

(e) Geometrically explain your answers to parts (a) and (c).

5. Let $\vec{F}(x, y, z) = \langle 6xyz, 3x^2z, 3x^2y + 2z \rangle$.

(a) Verify $\phi(x, y, z) = 3x^2\, y\, z + z^2$ is a potential for $\vec{F}$.

(b) Find $\int_C \vec{F} \cdot d\vec{r}$, where C is given by: $\vec{r}(t) = \langle \sin(\pi t) - 1, t^2 + 1, \cos(\pi t) \rangle$, $0 \le t \le 1$.

HINT: Before you dive into this integral, remember... you have a potential for $\vec{F}$.

6. Find $\int_C 3y\,dx + 3x\,dy + 3z\,dz$ where C is the path parametrized by:

$$\vec{r}(t) = \left\langle \frac{4t}{t^2+1}, \arctan(t), \log(10-9t) \right\rangle, \quad 0 \le t \le 1$$

HINT: View this integral as work integral and and check to see if the underlying field is conservative . . .

7. Let C be the triangle with vertices $(0,0)$, $(3,0)$, and $(0,4)$, oriented counter-clockwise.

   Find the (outward) flux of $\vec{F}(x,y) = \left\langle \frac{x}{2}, \frac{y}{2} \right\rangle$ across C using Green's Theorem.

   Compare your answer to what you obtained on # 4.

8. Use Green's Theorem to evaluate the line integral $\int_C x\,dy - y\,dx$ where C is the three part path:

- part one: the line segment from $(0, 0)$ to $(\sqrt{2}, -\sqrt{2})$
- part two: the circular arc $x^2 + y^2 = 4$ from $(\sqrt{2}, -\sqrt{2})$ to $(\sqrt{2}, \sqrt{2})$
- part three: the line segment from $(\sqrt{2}, \sqrt{2})$ to $(0, 0)$

9. Let $\vec{F}(x, y) = \langle M(x, y), N(x, y) \rangle = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$.

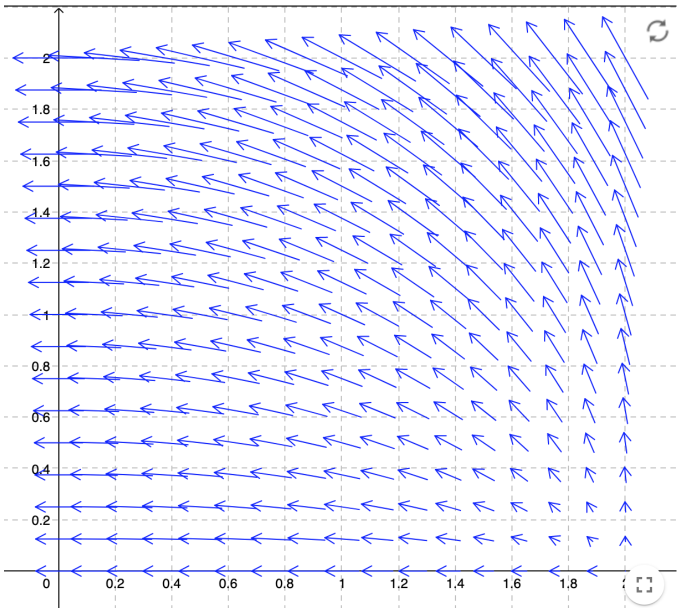
(a) Show $M_y(x, y) = N_x(x, y)$

(b) Let C be the Unit Circle, oriented counter-clockwise. Show $\oint_C \vec{F} \cdot d\vec{r} \neq 0$.

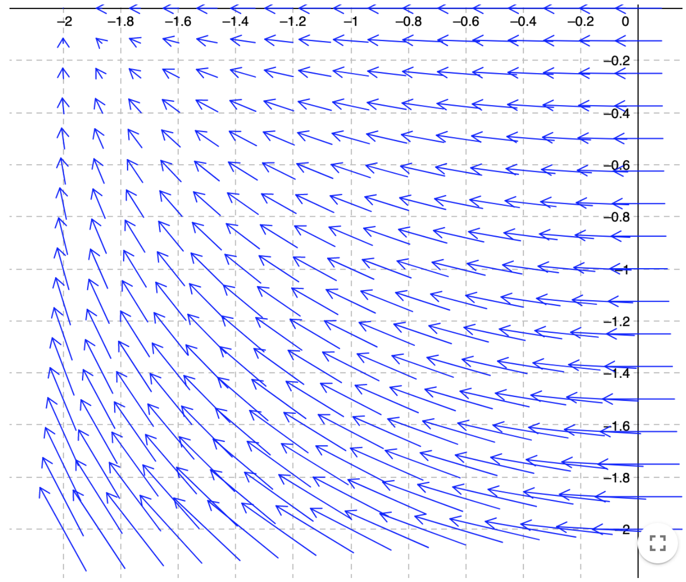
HINT: Parametrize the Unit Circle as: $x = \cos(t)$, $y = \sin(t)$, $0 \leq t \leq 2\pi$.

(c) Is $\vec{F}$ conservative? Explain.

10. Below is the graph of a vector field $\vec{F}$ near $(x, y) = (1, 1)$ and $(x, y) = (-1, -1)$:



near $(x, y) = (1, 1)$



near $(x, y) = (-1, -1)$

(a) Use the graph to explain why both the divergence and scalar curl of $\vec{F}$ at $(1, 1)$ are positive

(b) Use the graph to explain why both the divergence and scalar curl of $\vec{F}$ at $(-1, -1)$ are negative.

(c) Is $\vec{F}$ conservative? Is $\vec{F}$ divergence free? Explain.

11. Let S be given by: $\vec{r}(u, v) = \left\langle u, u^2 + 4v^2, 2v \right\rangle$, $-4 \leq u \leq 4$, $-2 \leq v \leq 2$.

(a) Convert the given representation to rectangular coordinates to sketch or otherwise describe the graph.

(b) Find an expression for $\vec{r}_u \times \vec{r}_v$.

(c) Write the equation of the tangent plane to this surface at the point $(-1, 5, 2)$.

(d) Set up, but do not evaluate, an integral which determines the area of S.

12. Let S be the portion of the plane $4x + 2y + z = 4$ which lies in the first octant.

Suppose the density of S is given by $\rho(x, y, z) = z$ with units of $\dfrac{\text{mass}}{\text{area}}$.

Find and **interpret** $\displaystyle\iint_S \rho(x, y, z)\, dS$ by evaluating a double integral with order $dy\, dx$.

13. Let S be given by: $\vec{r}(u,v) =< 2\sin(u)\cos(v), 2\sin(u)\sin(v), 2\cos(u) >$ for $\frac{\pi}{4} \leq u \leq \frac{3\pi}{4}$ and $0 \leq v \leq 2\pi$.

(a) Find and simplify $\vec{r}_u \times \vec{r}_v$

HINT: Be on the lookout for 'Pythagorean Magic.'

(b) Use a double integral to compute the flux of $\vec{F}(x, y, z) = < x, y, z >$ across S using $\vec{n} = \vec{r}_u \times \vec{r}_v$.

14. Let $\vec{F}(x, y, z) = \langle y, x, z^2 \rangle$. Let Q be the solid bounded by $z = 4$ and $z = x^2 + y^2$ with boundary S.

(a) Sketch Q and the projection of Q in the xy-plane.

(b) Use the Divergence Theorem to find the outward flux of $\vec{F}$ across S by computing:

$$\iiint_Q \nabla \cdot \vec{F} \, dV$$

HINT: This is best evaluated using cylindrical coordinates.

15. Let $\vec{F}(x, y, z) = \langle 3z, 2y, x \rangle$. Let S be the portion of $x + 3y + 2z = 12$ which lies in the first octant.

Let C be the boundary of S oriented by the upward pointing normal.

(a) Sketch S and its projection into the xy-plane.

(b) Use Stokes's Theorem to find the work done moving a particle along C by computing:

$$\int\int_S (\nabla \times \vec{F}) \cdot \hat{N}_{\text{up}} \, dS$$

HINT: Feel free to use properties of double integrals to simplify your computation!

16. Let $\vec{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$ where M, N, and P have continuous second partials.

Show: $\nabla \cdot \left(\nabla \times \vec{F}\right) = 0$.

NOTE: Be sure to explain why the continuity of the component functions' second partials is important!

HINT: Anyone remember the triple scalar product?

17. The key notion of Calculus is the limit. Recall the following definitions:

- For a **real-valued** function f and **real numbers** a and L, $\lim_{x \to a} f(x) = L$ means:

  given $\epsilon > 0$, there is a $\delta > 0$ so that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

- For a **vector-valued** function $\vec{r}$ and **real number** a and **vector** $\vec{L}$, $\lim_{t \to a} \vec{r}(t) = \vec{L}$ means:

  given $\epsilon > 0$, there is a $\delta > 0$ so that if $0 < |t - a| < \delta$, then $\|\vec{r}(t) - \vec{L}\| < \epsilon$.

- For a **real-valued** function f and **point** (a, b) and real number L, $\lim_{(x,y) \to (a,b)} f(x, y) = L$ means:

  given $\epsilon > 0$, there is a $\delta > 0$ so that if $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$, then $|f(x, y) - L| < \epsilon$.

(a) Using the broad ideas of 'inputs,' 'outputs', and 'closeness,' write a sentence (or two!) explaining how all of these definitions are getting at the same idea.

(b) Come up with your own $\epsilon - \delta$ definition for what it means for $\lim_{(x,y) \to (a,b)} \vec{F}(x, y) = \vec{L}$

18. An important property of functions is **continuity**. Recall the following definitions:

- For a **real-valued** function f and **real number** a, f is continuous at a means: $\lim_{x \to a} f(x) = f(a)$.
- For a **vector-valued** function $\vec{r}$ and **real number** a, $\vec{r}$ is continuous at a means $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$.
- For a **real-valued** function f and **point** (a, b), f is continuous at (a, b) means $\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$.

(a) Using the broad ideas of 'inputs,' 'outputs', and 'closeness,' write a sentence (or two!) explaining how all of these definitions are getting at the same idea.

(b) Come up with your own definition for what it means for a vector field $\vec{F}$ to be continuous at (a, b).

19. An important operation we perform on functions is **differentiation**. Recall the following definitions:

Assuming the limits below exist ...

- For a **real-valued** function f and **real number** a, $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.
- For a **vector-valued** function $\vec{r}$ and **real number** a, $\vec{r}\,'(a) = \lim_{h \to 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h}$.
- For a **real-valued** function f and **point** (a, b), $f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$.

(a) Using the broad ideas of 'inputs,' 'outputs', and 'rate of change,' write a sentence (or two!) explaining how all of these definitions are getting at the same idea.

(b) Come up with your own limit definition for $\vec{r}_u(a, b)$.

20. Geometrically, ‘differentiability’ means ‘local linearity.’ As we saw with functions of two variables, just having the partial derivatives exist was not enough to guarantee local linearity. Recall the following definition:

f is differentiable at $(x, y) = (a, b)$ means there are functions $\epsilon_1(x, y)$ and $\epsilon_2(x, y)$ so that:

$$f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + \epsilon_1(x, y)(x - a) + \epsilon_2(x, y)(y - b),$$

where $\lim_{(x,y)\to(a,b)} \epsilon_1(x, y) = 0$ and $\lim_{(x,y)\to(a,b)} \epsilon_2(x, y) = 0$.

(a) Write a sentence (or two!) which explains why, using the above definition, if f is differentiable at $(x, y) = (a, b)$ then if (x, y) is ‘close to’ (a, b), the surface $z = f(x, y)$ can be closely approximated by the tangent plane at $(a, b, f(a, b))$.

(b) Suppose $w = w(x, y, z, t)$ is a function of **four** variables. Write your own definition as to what it means for w to be differentiable at a point $(x, y, z, t) = (a, b, c, d)$.

21. **Integration** can be thought of as 'continuous summation' or 'accumulation.' The Fundamental Theorem of Calculus connects integration and differentiation as follows. Assuming f' is continuous:

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

Recently, we've seen the following formulas:

$$\int_C \nabla f \cdot d\vec{r} = f(\text{ending point of } C) - f(\text{starting point of } C)$$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{N}\,dS = \oint_C \vec{F} \cdot d\vec{r} \qquad \text{and} \qquad \iiint_Q \nabla \cdot \vec{F}\,dV = \oiint_S \vec{F} \cdot \hat{N}\,dS$$

Using the concepts of 'rate of change,' 'accumulation,' and 'oriented boundary,' explain how these formulas are all getting at the same idea.