MATH 2700: TAKE HOME 03 (50 points.)

NAME: ______________________________

DUE: The day of Test 3, at the beginning of class.

DIRECTIONS: Show all work.

1. Let $f(x,y) = \dfrac{4xy^2}{x^2+y^2}$.

   (a) Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ along the path $y = 2x$.

   (b) Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ along the path $y = x^2$.

(c) What, if anything, can be said about $\lim_{(x,y)\to(0,0)} f(x,y)$ based on your answers to (a) and (b)? Explain.

(d) Convert to polar coordinates: $x = r\cos(\theta)$ and $y = r\sin(\theta)$ to find $\lim_{(x,y)\to(0,0)} f(x,y)$.

HINT: Remember: $(x,y) \to (0,0)$ is equivalent to $r \to 0$.

2. Let $f(x, y) = \dfrac{4x^2}{x^2 + y^2}$.

(a) Find $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ along the path $y = x$.

(b) Find $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ along the path $y = x^2$.

(c) Based on your answers to (a) and (b), does $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ exist? Explain.

3. Recall the **lift** of a plane curve given by $\vec{r}(t) = \langle x(t), y(t) \rangle$ to the surface described by $z = f(x, y)$ is

$$\vec{L_r}(t) = \langle x(t), y(t), f(x(t), y(t)) \rangle$$

Let $f(x, y) = \dfrac{4x^2}{x^2 + y^2}$. In #2, you investigated $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ along the paths $y = x$ and $y = x^2$.

(a) Find the lifts of the paths $y = x$ and $y = x^2$ to the graph of $f(x, y) = \dfrac{4x^2}{x^2 + y^2}$.

Lift of $y = x$ to the graph of f:

Lift of $y = x^2$ to the graph of f:

(b) Use a graphing utility to graph $z = f(x, y)$ along with your answers to (a) to check #2,

4. Consider the limit: $\displaystyle\lim_{(x,y)\to(0,0)} 3y^2 \sin\left(\frac{2x}{x^2+y^2}\right)$

(a) Explain why we cannot use direct substitution to evaluate this limit.

(b) Use the Squeeze Theorem to evaluate this limit.

HINT: Remember: $-1 \le \sin(\theta) \le 1$ for all angles θ.

5. Answer the following questions regarding the $\epsilon - \delta$ proof given below.

Proof that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$:

Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{4}$. If $0 < \sqrt{x^2+y^2} < \delta$, then:

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| = \left|\frac{4xy^2}{x^2+y^2}\right| = \frac{\left|4xy^2\right|}{\left|x^2+y^2\right|} = \frac{4|x|y^2}{x^2+y^2}$$

Question: why can we drop the absolute values on the 4, y^2 and x^2+y^2?

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| = \ldots = \frac{4|x|y^2}{x^2+y^2} = 4|x|\,\frac{y^2}{x^2+y^2} \le 4|x|$$

Question: what happened to the factor of $\frac{y^2}{x^2+y^2}$?

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| = \ldots \le 4|x| \le 4\sqrt{x^2+y^2}$$

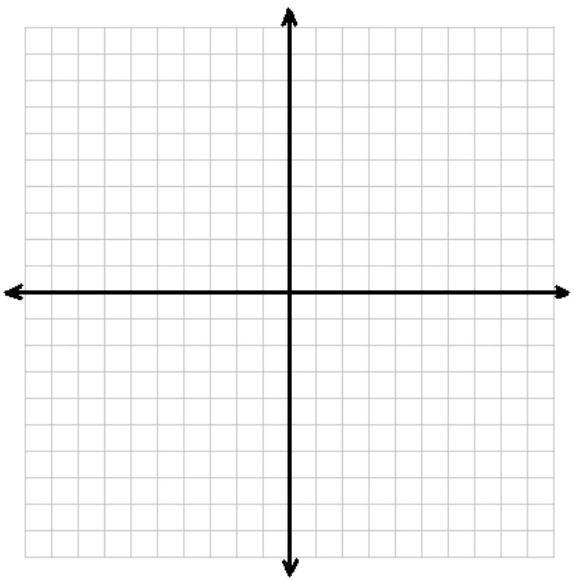
Question: why is $|x| \le \sqrt{x^2+y^2}$?

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| = \ldots \le 4\sqrt{x^2+y^2} < 4\left(\frac{\epsilon}{4}\right) = \epsilon \checkmark$$

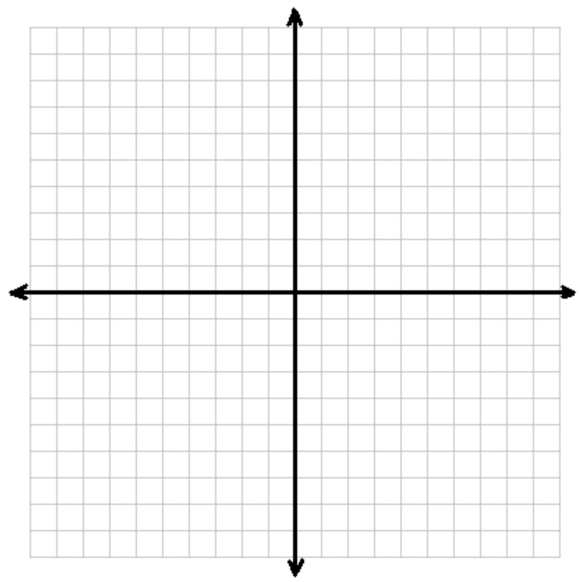
Question: why is $4\sqrt{x^2+y^2} < 4\left(\frac{\epsilon}{4}\right)$?

6. Let $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$.

(a) Sketch the domain of f in the xy-plane and sketch a contour map of f using the axes below.



Domain of f.



Contour map of f.

(b) Find and simplify $f_x(x, y)$ and $f_y(x, y)$.

(c) Show $f_{xy}(x, y) = f_{yx}(x, y)$.

7. Let $f(x, y) = x\, e^{\sin(\pi\, xy)}$

(a) Find and simplify:

i. $f_x(x, y)$

ii. $f_y(x, y)$

(b) Use f_x and f_y to show f is differentiable.

(c) Find and simplify the equation of the tangent plane to $z = f(x, y)$ at $(x, y) = \left(\frac{1}{2}, 0\right)$.

(d) Use your tangent plane to approximate $f(0.49, -0.01)$ and compare your answer to $f(0.49, -0.01)$

8. Let: $f(x,y) = \begin{cases} \dfrac{3x^2y^2}{x^4+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

(a) Determine: $\lim_{(x,y)\to(0,0)} f(x,y)$ along the paths $y = mx$.

(b) Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? Explain.

(c) Find $f_x(0,0)$ and $f_y(0,0)$ using the limit definition as shown in class.

(d) Is f continuous at $(0,0)$? Is f differentiable at $(0,0)$? Explain your reasoning.

9. The period of a pendulum T is a function of the length of its arm ℓ and the acceleration due to gravity, g:

$$T = 2\pi\sqrt{\frac{\ell}{g}} = 2\pi\ell^{\frac{1}{2}}\,g^{-\frac{1}{2}}$$

For the problems below, round your answers to <u>four</u> decimal places.

(a) Calculate the period of a pendulum T if $\ell = 4.4$ m and $g = 9.8\ \frac{\text{m}}{\text{s}^2}$.

(b) Find an expression for the differential dT in terms of ℓ, g, $d\ell$ and dg.

(c) If the measurements for ℓ and g are accurate to ± 0.05 m and $\pm 0.1\ \frac{\text{m}}{\text{s}^2}$, respectively, approximate the percent relative propagated error in calculating T using $\ell = 4.4$ m and $g = 9.8\ \frac{\text{m}}{\text{s}^2}$.

10. Use the definition of differentiability to prove $f(x, y) = 3xy$ is differentiable at all points in the plane.

11. Suppose $w = F(x, y, z)$ where $x = \rho\cos(\theta)\sin(\phi)$, $y = \rho\sin(\theta)\sin(\phi)$, and $z = \rho\cos(\phi)$

(a) Draw a tree of dependence connecting w with ρ, θ, and ϕ.

(b) Use the Chain Rule to show $\dfrac{\partial w}{\partial \theta} = x\,\dfrac{\partial w}{\partial y} - y\,\dfrac{\partial w}{\partial x}$.

12. Let c be a positive constant. Show $U(x,t) = e^{-t}\sin(c\,x)$ is a solution to the Heat Equation:

$$c^2 \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

13. The one-dimensional time-dependent *Wave Equation* is the partial differential equation:

$$u_{tt}(x,t) = c^2 u_{xx}(x,t),$$

where c is the constant speed of the wave. Show that if f and g have continuous second derivatives, then any function of the form $u(x,t) = f(x-ct) + g(x+ct)$ satisfies the Wave Equation.

14. In class we used the chain rule to show that if $f(x, y) = $ constant is the level curve of a differentiable function f which implicitly describes y as a function of x, then

$$\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}.$$

Suppose $F(x, y, z) = $ constant is the level surface of a differentiable function F which implicitly describes z as a function of x and y. Using the formula for $\frac{dy}{dx}$ above as a guide, make an educated guess as to a formula for $\frac{\partial z}{\partial x}$ in terms of $F_x(x, y, z)$ and $F_z(x, y, z)$:

$$\frac{\partial z}{\partial x} =$$

Consider the surface: $xz^3 = y^3 z + 8$. Use your formula to find $\frac{\partial z}{\partial x}$ implicitly.

HINT: Don't forget to rewrite the equation in the form $F(x, y, z) = $ constant.

15. Let $f(x,y) = \ln\left|2x^2 - y^2\right|$.

(a) Find $f_x(x,y)$ and $f_y(x,y)$ and use these to show f is differentiable at $(1,2)$.

(b) Find $\nabla f(1,2)$.

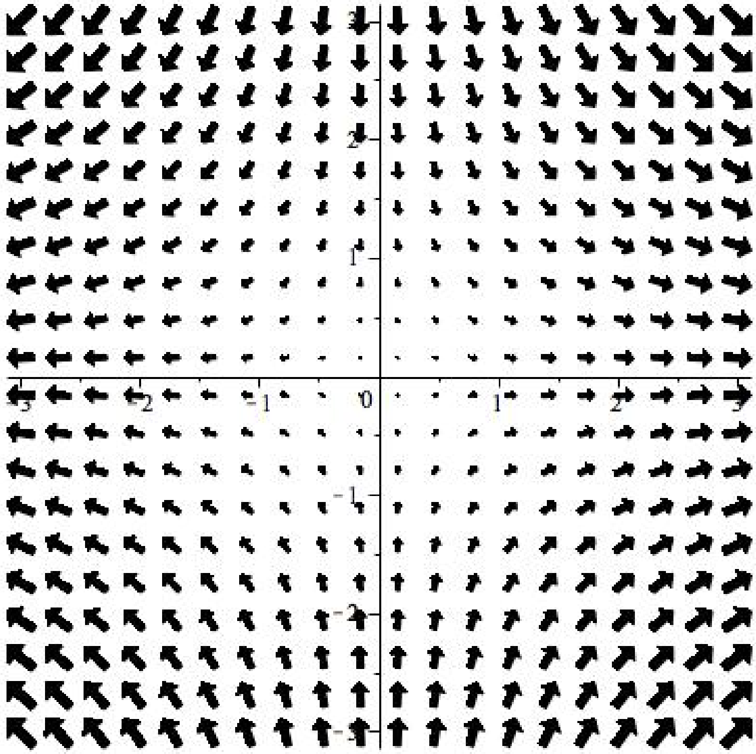
(c) What is the maximum value of $D_{\hat{u}} f(1,2)$? What is $\hat{u}$ in this case?

(d) What is the minimum value of $D_{\hat{u}} f(1,2)$? What is $\hat{u}$ in this case?

16. Find the equation of the tangent plane for the surface $xy\cos(z) = 1$ at the point $\left(4, -\dfrac{1}{2}, \dfrac{2\pi}{3}\right)$.

Check your answer using a graphing utility.

17. Below is a sketch of $\nabla f(x, y)$. Explain what kind of phenomenon (local maximum, local minimum or saddle point) is occurring at $(0, 0)$. Explain your reasoning.



18. Suppose f has continuous partial derivatives of all orders everywhere in the plane. Suppose:

$f_x(-1,2) = 0$ $\quad$ $f_y(-1,2) = 0$ $\quad$ $f_{xx}(-1,2) = 3$ $\quad$ $f_{yy}(-1,2) = 5$

$$f_{xy}(-1,2) = f_{yx}(-1,2) = 2$$

What feature does the graph have at $(-1, 2, f(-1,2))$? Explain your reasoning.

19. Suppose f has continuous partial derivatives of all orders everywhere in the plane. Suppose:

$f_x(-1,2) = 0$ $\quad$ $f_y(-1,2) = 0$ $\quad$ $f_{xx}(-1,2) = 3$ $\quad$ $f_{yy}(-1,2) = -5$

What feature does the graph have at $(-1, 2, f(-1,2))$? Explain your reasoning.

20. Find the local maximums, minimums, and saddle points of $z = f(x, y) = 27x^3 - y^3 - 9xy$.

21. Here's another example where the 'Lone Critical Value' principle from Calc 1 doesn't carry over to Calc 3.

Let $f(x, y) = x^2 + y^2(1 - x)^3$.

(a) Show f has only one critical point in the plane, and show it has a local minimum there.

What is the local minimum value?

(b) Find $f(2, 3)$. Explain how this shows the local minimum in part (a) cannot be an absolute minimum.

Graph f using a graphing utility to check your answers.

22. Recall the Cobb-Douglas Production Model we've been working on in class this chapter:

$$Q(x, y) = 150x^{1/3}y^{2/3}, \quad x > 0, \quad y > 0$$

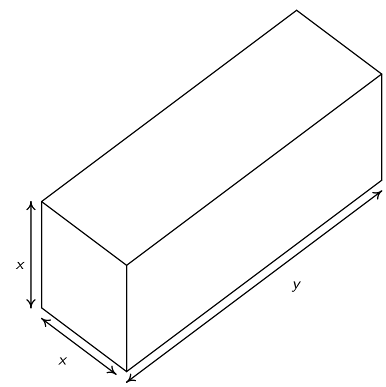
Here Q is the amount of units produced annually, in *thousands*, and x and y represent the amount of money, in *millions of dollars*, spent annually on labor and capital, respectively. Currently, \$1 million is spent on labor and \$8 million is spent on capital.

If an additional \$40 million is available to invest in the company, use the Method of Lagrange Multipliers to determine how best to invest these additional funds in order to maximize production. How does your answer compare to the estimates we've determined in class?

HINT: With \$40 million additional dollars, you can set the entire budget equal to \$49 million.

That is, maximize $Q(x, y) = 150x^{1/3}y^{2/3}$ subject to $x + y = 49$.

23. Per US Postal regulations, the dimensions of a rectangular box which is to be shipped using 'parcel post' must satisfy the inequality 'Length $+$ Girth ≤ 130 inches.' Let's assume we have a closed rectangular box with a square face of side length x inches as drawn below. Let y be the length of the box, also measured in inches. The girth is the distance around the box in the other two dimensions so in our case it is the sum of the four sides of the square which in this case is $4x$. Find the dimensions of the box of maximum volume that can be shipped via Parcel Post. Find exact answers then round your answers to two decimal places.



24. (a) Use Lagrange Multipliers to minimize $F(x, y, z) = x^2 + y^2 + z^2$ subject to $Ax + By + Cz = D$.

HINT: Solve for x, y, z in terms of λ then solve for λ using the constraint equation. Exploit symmetry.

(b) Geometrically interpret your answer to part (a).

QUADRIC SURFACE EXPLORATION: DISCRIMINANTS REVISITED

Use a graphing utility to investigate surfaces of the form: $f(x,y) = A\,x^2 + B\,xy + C\,y^2$. Let $D = 4AC - B^2$.

In each case below, record the value of A and D and describe the shape of the graph of f near the origin.

$f(x,y) = x^2 + y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = -x^2 + y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = x^2 - y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = x^2 + xy + y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = x^2 + 2xy + y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = x^2 + 3xy + y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = x^2 - xy + y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = x^2 - 2xy + y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = x^2 - 3xy + y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = x^2 + xy - y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = x^2 + 2xy - y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = x^2 + 3xy - y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = x^2 - xy - y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = x^2 - 2xy - y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = x^2 - 3xy - y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = -x^2 + xy - y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = -x^2 + 2xy - y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = -x^2 + 3xy - y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = -x^2 - xy - y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = -x^2 - 2xy - y^2$	$A =$ ______	$D =$ ______	Shape?
$f(x,y) = -x^2 - 3xy - y^2$	$A =$ ______	$D =$ ______	Shape?

Make a conjecture about the **signs** of A and D as related to the shape of the graph near the origin.

SECOND PARTIALS TEST EXPLORATION: TAYLOR SERIES REVISITED

In single variable Calculus, the notions of 'local linearity' and 'tangent line' were generalized (literally to the n th degree!) with the notion of Taylor Polynomials. In particular, recall the second degree Taylor Polynomial for a twice differentiable function f centered at $x = a$, $T_2(x)$, is defined as:

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2.$$

In this case, the parabola $y = T_2(x)$ is the best fit quadratic function to the graph of $y = f(x)$ at $(a, f(a))$ and provides another way to see why $f''(a)$ determines the concavity of the graph of f at $x = a$.
In much the same way, we can generalize the notions of local linearity and tangent **planes** in multivariable Calculus using Taylor Polynomials. The second degree Taylor Polynomial centered at $(x, y) = (a, b)$ is[1]

$$\begin{aligned} T_2(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \ldots \\ &\ldots + \frac{1}{2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + fyy(a,b)(y-b)^2\right] \end{aligned}$$

- Let A, B, and C be the coefficients of x^2, xy, and y^2, respectively in the generic formula for $T_2(x, y)$.

  Determine formulas for A, B, and C in terms of $f_{xx}(a, b)$, $f_{xy}(a, b)$, and $f_{yy}(a, b)$.

  Express $D = 4AC - B^2$ in terms of $f_{xx}(a, b)$, $f_{yy}(a, b)$ and $f_{xy}(a, b)$.

- Find $T_2(x, y)$ for $f(x, y) = x^4 + y^4 - 4xy + 2$ centered at $(0, 0)$ and calculate D.

  Graph f and T_2 near $(x, y) = (0, 0)$ and compare both graphs.

- Repeat part (b) but move the center to $(1, 1)$.

- How does all this connect to the previous page's observations and the Second Partial's Test?

[1] This is assuming $f_{xy}(a, b) = f_{yx}(a, b)$. If this is not true, we'd have both $f_{xy}(a, b)(x - a)(y - b)$ and $f_{yx}(a, b)(x - a)(y - b)$ terms.

VERIFYING LAGRANGE'S THEOREM ON A MONKEY SADDLE

Let $f(x, y) = x^3 - 3xy^2$. The graph of this function is called a 'Monkey Saddle.'

1. Graph $z = f(x, y)$ using a graphing utility. Why do you think the graph is called a Monkey Saddle?

2. Suppose x and y are differentiable functions of t.

   Use the multivariable chain rule to find an expression for $f'(t)$ involving x, y, $x'(t)$ and $y'(t)$.

3. Consider the Unit Circle: $g(x, y) = x^2 + y^2 = 1$ as parameterized: $\vec{r}(t) = \langle \cos(t)\ \sin(t) \rangle$, $0 \leq t < 2\pi$.

   Use part 2 to help you find the highest and lowest points (x, y, z) on the lift of $\vec{r}$ to the graph of f.

   HINT: Start by analyzing $f'(t)$ ...

   REMEMBER: Your final answer should be **points** of the form (x, y, z).

4. Verify Lagrange's Theorem at each of the points you found in part 3.

   That is, show $\nabla f(a, b) = \lambda \nabla g(a, b)$ for some real number λ at those points.