MATH 2700: TEST 02 (100 points.)

NAME: ______________________________

DIRECTIONS: Show all work.

1. Let $\vec{r}(t) = \left\langle \sqrt{t+4}, \cos(\pi t), \ln(1-t) \right\rangle$.

   (a) List the interval(s) of continuity of $\vec{r}$.

   (b) Find $\vec{r}(0)$ and $\vec{r}\,'(0)$.

   (c) Use part (b) to find an equation of the tangent line to the curve traced out by $\vec{r}$ at $(2, 1, 0)$.

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2. Suppose the acceleration of an object is given by: $\vec{a}(t) = \langle -18\cos(3t), -18\sin(3t)\rangle$.

(a) If the initial velocity of the object is $\langle 0, 4\rangle$, find an expression for the velocity, $\vec{v}(t)$.

(b) If the initial position of the object is $\langle 5, 0\rangle$, find an expression for the position, $\vec{r}(t)$.

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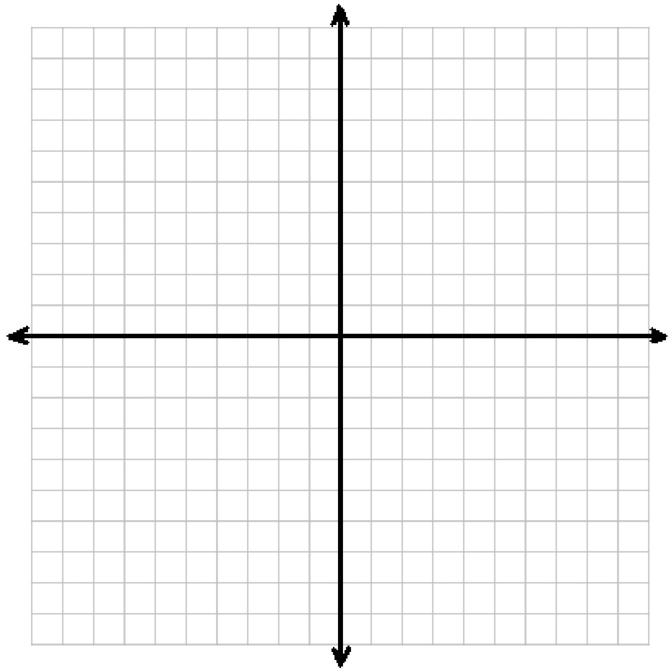
3. Let $\vec{r}(t) = \left\langle t, 4 - t^2 \right\rangle$.

(a) Graph $\vec{r}$. Use each square as half a unit. Be sure to include orientation.



(b) Find and simplify: $\vec{v}(t)$, $\|\vec{v}(t)\|$, and $\vec{a}(t)$.

(c) Find and simplify: $\vec{v}(t) \cdot \vec{a}(t)$ and $|\vec{v}(t) \cdot \vec{a}(t)|$.

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(d) Find and simplify $\vec{v}(t) \times \vec{a}(t)$ and $\|\vec{v}(t) \times \vec{a}(t)\|$.

NOTE: View $\vec{v}(t)$ and $\vec{a}(t)$ as 3D by assigning them both a z-component of 0.

(e) Show that $\hat{T}(t) = \left\langle \frac{1}{\sqrt{4t^2+1}}, -\frac{2t}{\sqrt{4t^2+1}} \right\rangle$.

(f) Use the fact that the motion is 2-D to list the two candidates for $\hat{N}(t)$:

$\hat{N}(t) =$ or $\hat{N}(t) =$

(g) Use your graph from part (a) to determine $\hat{N}(0)$. Use $\hat{N}(0)$ to select the correct formula for $\hat{N}(t)$.

$\hat{N}(0) =$ so $\hat{N}(t) =$

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(h) Find and simplify: $a_T(t)$, and $a_N(t)$ and verify $\vec{a}(t) = a_T(t)\hat{T}(t) + a_N(t)\hat{N}(t)$.

- $a_T(t) =$

- $a_T(t)\hat{T}(t) =$

- $a_N(t) =$

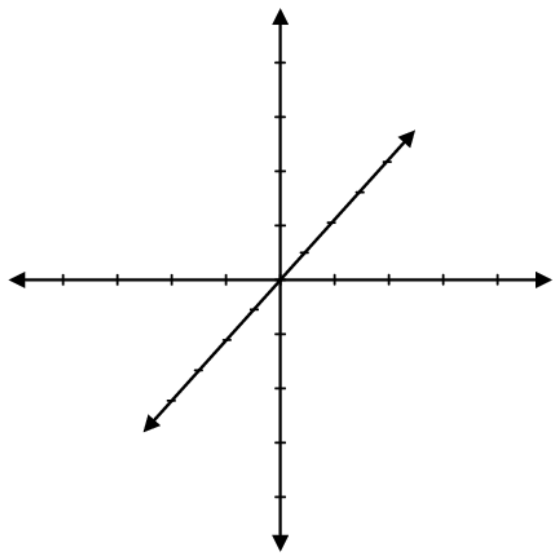
- $a_N(t)\hat{N}(t) =$

- $a_T(t)\hat{T}(t) + a_N(t)\hat{N}(t) =$

4. Let $\vec{r}(t) = \langle 4\cos(t), 3t, 4\sin(t) \rangle$, $t \geq 0$.

(a) Show the graph of $\vec{r}$ lies on the cylinder $x^2 + z^2 = 16$.

(b) Sketch or otherwise describe the graph of $\vec{r}(t)$.



(c) Find the arc length of the graph of $\vec{r}(t)$ from $(4, 0, 0)$ to $(-4, 3\pi, 0)$.

5. For $\vec{r}(t) = \langle \cos(t), \sin(t), \cos(2t) \rangle$, $\hat{T}\left(-\frac{\pi}{4}\right) = \left\langle \frac{\sqrt{10}}{10}, \frac{\sqrt{10}}{10}, \frac{2\sqrt{5}}{5} \right\rangle$ and $\hat{N}\left(-\frac{\pi}{4}\right) = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right\rangle$.

   Find the equation of the osculating plane of the graph of $\vec{r}$ corresponding to the point when $t = -\frac{\pi}{4}$.

   Write your answer in the form $Ax + By + Cz = D$ where $A > 0$.

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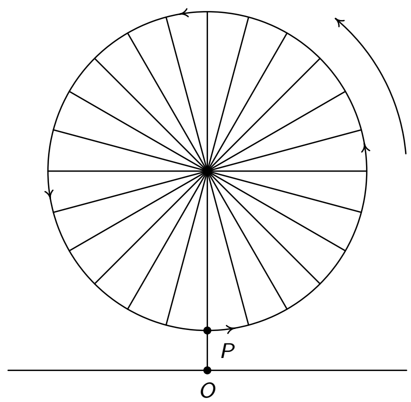
6. The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform making its overall height 136 feet. It completes two revolutions in 2 minutes and 7 seconds. Assume that the passengers are at the edge of the circle and the rotation is counter-clockwise (otherwise we could just observe the ride from the other side.)



(a) Find a vector valued function, $\vec{r}_{\mathrm{W}}(t) = \langle R\cos(\omega t), R\sin(\omega t)\rangle$ which models the passengers' path around the center of the wheel.

HINT: The wheel makes two **revolutions** (4π radians) in 2 minutes, 7 seconds (127 seconds) ...

(b) Adjust your answer to part (a) to account for the 8 foot tall platform so that the y-component models the actual height of the passengers off of the ground.

(c) Adjust your answer to part (b) so that $t = 0$ corresponds to the point P on the diagram - the point on the wheel closest to the platform.

7. Recall one of the formulas for the curvature of the graph of a vector-valued function is: $\kappa = \dfrac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$.

Find and simplify the curvature of $\vec{r}(t) = \langle R\cos(\omega t), R\sin(\omega t)\rangle$. Assume $R, \omega > 0$.

NOTE: View $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$ as 3D by assigning them both a z-component of 0 to compute $\vec{r}\,'(t) \times \vec{r}\,''(t)$.

What is the radius of curvature in this case? Why does that make sense?

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BONUS: Use the formula below to show that if the curvature of a plane curve $y = f(x)$ is 0, then f is linear.

$$\kappa = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}$$

$\overline{}$ $+5$

CURVE ANALYSIS FORMULA SUMMARY SHEET

- **POSITION:** $\vec{r}$
- **VELOCITY:** $\vec{v} = \vec{r}\,'$
- **ACCELERATION:** $\vec{a} = \vec{v}\,'$
- **PRINCIPAL UNIT TANGENT VECTOR** $\hat{T} = \dfrac{\vec{v}}{\|\vec{v}\|}$
- **PRINCIPAL UNIT NORMAL VECTOR:** $\hat{N} = \dfrac{\hat{T}\,'}{\|\hat{T}\,'\|}$
- **COMPONENTS OF ACCELERATION:** $\vec{a} = a_T\,\hat{T} + a_N\hat{N}$ where $a_T = \dfrac{\vec{v}\cdot\vec{a}}{\|\vec{v}\|}$ and $a_N = \dfrac{\|\vec{v}\times\vec{a}\|}{\|\vec{v}\|}$
- **PRINCIPAL BINORMAL VECTOR:** $\hat{B} = \hat{T}\times\hat{N} = \dfrac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|}$
- **CURVATURE:** $\kappa = \left\|\dfrac{d\,\hat{T}}{ds}\right\| = \dfrac{d\,\hat{T}}{ds}\cdot\hat{N} = \dfrac{\|\vec{v}\times\vec{a}\|}{\|\vec{v}\|^3}$

  NOTE: $\dfrac{d\,\hat{T}}{ds} = \kappa\hat{N}$
- **TORSION:** $\tau = -\dfrac{d\hat{B}}{ds}\cdot\hat{N} = \dfrac{(\vec{v}\times\vec{a})\cdot\vec{a}\,'}{\|\vec{v}\times\vec{a}\|^2}$.

  NOTE: $\dfrac{d\hat{B}}{ds} = -\tau\hat{N}$
- **FRENET - SERRET EQUATIONS:**

  From $\dfrac{d\,\hat{T}}{ds} = \kappa\hat{N}$ and $\dfrac{d\hat{B}}{ds} = -\tau\hat{N}$, we can differentiate $\hat{N} = \hat{B}\times\hat{T}$, to get:

$$\begin{cases} \dfrac{d\,\hat{T}}{ds} & = & \kappa\,\hat{N} \\[2ex] \dfrac{d\hat{N}}{ds} & = & -\kappa\,\hat{T} + \tau\,\hat{B} \\[2ex] \dfrac{d\hat{B}}{ds} & = & -\tau\,\hat{N} \end{cases}$$