MATH 2700: TAKE HOME 02 (50 points.)

NAME: ______________________________

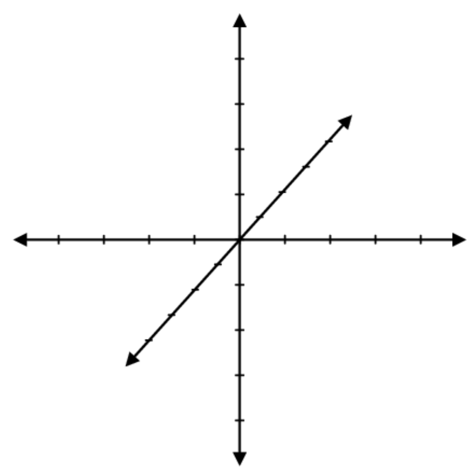
DUE: The day of Test 2, at the beginning of class.

DIRECTIONS: Show all work.

1. Let $\vec{r}(t) = \langle 5\cos(t), 5\sin(t), 12t \rangle,\ t \geq 0$.

   (a) Show the graph of $\vec{r}$ lies on the cylinder: $x^2 + y^2 = 25$.

   (b) Sketch or otherwise describe the graph of $\vec{r}(t)$.



   (c) What is the name of this curve?

2. Assume the Earth orbits the Sun in 364.25 days in a circular path with radius 93,000,000 miles and the Moon orbits the Earth in 27.3 days in a circular path with radius 238,900 miles.

(a) Find a vector valued function, $\vec{r}_{\mathsf{E}}\,(t)$ which models Earth's orbit around the Sun.

RECALL: Circular motion can be modeled by: $\vec{r}(t) = \langle R\cos(\omega t), R\sin(\omega t)\rangle$ where $R > 0$ is the radius of the circle and $\omega > 0$ is the **angular** frequency.[1]

(b) Find a vector valued function, $\vec{r}_{\mathsf{M}}\,(t)$ which models Moon's orbit around the Earth.

(c) What does $\vec{r}_{\mathsf{E}}\,(t) + \vec{r}_{\mathsf{M}}\,(t)$ model? Try graphing $\vec{r}_{\mathsf{E}}\,(t) + \vec{r}_{\mathsf{M}}\,(t)$. What difficulties do you encounter?

[1]. . . units are $\dfrac{\text{radians}}{\text{time}}$

3. The helical wind turbine at Progressive Field is 18 feet in diameter, 40 feet tall, and makes 3.5 turns from bottom to top (See picture.) Find a vector-valued function which traces out the helix on this turbine. What assumptions do you have to make?

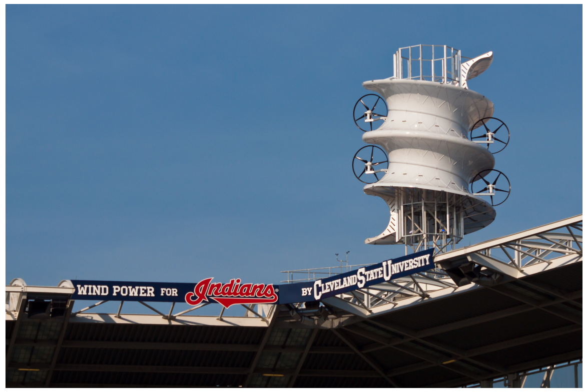


Image taken from: Flikr courtesy of Christopher Irwin.

4. Let $\vec{r}(t) = \langle \cos(t), \sin(t), \cos(2t) \rangle$, $0 \leq t \leq 2\pi$.

(a) Show the graph of $\vec{r}$ lies on the surface $z = x^2 - y^2$. Check your answer using a graphing utility.

HINT: $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ ...

(b) Find $\vec{r}\left(\frac{3\pi}{4}\right)$ and $\vec{r}\,'\left(\frac{3\pi}{4}\right)$.

(c) Find a vector form of the equation of the tangent line to the graph of $\vec{r}$ when $t = \frac{3\pi}{4}$.

Check your answer using a graphing utility.

5. (a) Find the domain of $\vec{r}(t) = \left\langle \sqrt{1-t}, e^{-2t}, \dfrac{\sin(2t)}{3t} \right\rangle$.

(b) Find $\displaystyle\lim_{t \to 0^+} \left\langle \sqrt{1-t}, e^{-2t}, \frac{\sin(2t)}{3t} \right\rangle$.

(c) State the interval(s) of continuity for $\vec{r}(t) = \left\langle \sqrt{1-t}, e^{-2t}, \dfrac{\sin(2t)}{3t} \right\rangle$.

6. The acceleration of an object is given by $\vec{a}(t) = \langle -4\cos(2t), 4\sin(2t) \rangle$.

(a) If the initial velocity of the object is $\langle 2, 2 \rangle$, find an expression for the velocity of the object, $\vec{v}(t)$.

(b) If the initial position of the object is $\langle 1, -1 \rangle$, find an expression for the position of the object, $\vec{r}(t)$.

7. If a particle moves with mass m along a position vector $\vec{r}(t)$:

- The **angular momentum**, $\vec{L}$ is defined as $\vec{L}(t) = \vec{r}(t) \times (m\ \vec{v}(t))$
- the **torque**, $\vec{\tau}$ is given by $\vec{\tau}(t) = \vec{r}(t) \times (m\,\vec{a}(t))$.

(a) Fill in the reasoning below which proves $\vec{L}'(t) = \vec{\tau}(t)$.

STATEMENT: $\vec{L}'(t) = D_t\left[\vec{r}(t) \times (m\ \vec{v}(t))\right] = D_t\left[\vec{r}(t)\right] \times (m\ \vec{v}(t)) + \vec{r}(t) \times D_t\left[m\,\vec{v}(t)\right]$

REASONING:

STATEMENT: $D_t\left[\vec{r}(t)\right] \times (m\ \vec{v}(t)) = \vec{v}(t) \times (m\,\vec{v}(t)) = m\,(\vec{v}(t) \times \vec{v}(t)) = m\,\vec{0} = \vec{0}$.

REASONING:

STATEMENT: $\vec{r}(t) \times D_t\left[m\ \vec{v}(t)\right] = \vec{r}(t) \times (m\,D_t\left[\vec{v}(t)\right]) = \vec{r}(t) \times (m\,\vec{a}(t)) = \vec{\tau}(t)$.

REASONING:

CONCLUSION: Hence, $\vec{L}'(t) = \vec{0} + \vec{\tau}(t) = \vec{\tau}(t)$.

(b) The Law of Conservation of Angular Momentum states that angular momentum is conserved if there is no torque. Using part (a), prove this claim by showing that if $\vec{\tau}(t) = \vec{0}$, then $\vec{L}(t)$ is constant.

NOTE: For the physics fans out there: $m\,\vec{v}(t) = \vec{p}$, the particle's linear momentum, so $\vec{L} = \vec{r} \times \vec{p}$.

According to Newton's Second Law, the force acting on the particle is $\vec{F} = m\,\vec{a}(t)$.

Substituting $\vec{F}$ in for $m\,\vec{a}(t)$ gives us the formula for torque from the section on Cross Products: $\vec{\tau} = \vec{r} \times \vec{F}$.

8. Suppose $\vec{r}(t)$ is the position of Spaceman Floyd's Cosmic Cruiser, *The Tennessee II*. (Here, we take Earth to be the origin.) Match each verbal description with the corresponding mathematical quantity.

This is how far Spaceman Floyd is from Earth.	$a_T(t)$
This is how fast Spaceman Floyd is moving.	$\widehat{T}(t)$
This is the direction Spaceman Floyd is traveling.	$\|\vec{r}(t)\|$
This is how fast Spaceman Floyd is turning.	$\|\vec{v}(t)\|$
This is how fast Spaceman Floyd's speed is changing.	$\|\widehat{T}\,'(t)\|$

9. The so-called 'twisted cubic' is given by the function: $\vec{r}(t) = \left\langle t, \frac{t^2}{\sqrt{2}}, \frac{t^3}{3} \right\rangle$, $t \geq 0$.

(a) Find and **simplify** expressions for $\vec{v}(t)$, $\|\vec{v}(t)\|$, and $\vec{a}(t)$.

HINT: $t^4 + 2t^2 + 1 = (t^2 + 1)^2$ ...

(b) Find the arc length of the twisted cubic from $(0, 0, 0)$ to $\left(1, \frac{\sqrt{2}}{2}, \frac{1}{3}\right)$.

(c) Find and simplify expressions for $a_T(t)$ and $a_N(t)$.

(d) Find and simplify expressions for $\widehat{T}(t)$ and $\widehat{N}(t)$,

HINT: Once you get $\widehat{T}(t)$, consider solving $\vec{a}(t) = a_T(t)\widehat{T}(t) + a_N(t)\widehat{N}(t)$ for $\widehat{N}(t)$ ...

(e) Find and simplify an expression for $\widehat{B}(t)$.

HINT: You ostensibly found $\vec{v} \times \vec{a}$ and $\|\vec{v} \times \vec{a}\|$ already ...

(f) Find an equation of the osculating plane at $\left(1, \frac{\sqrt{2}}{2}, \frac{1}{3}\right)$.

Graph $\vec{r}$ near $\left(1, \frac{\sqrt{2}}{2}, \frac{1}{3}\right)$ along with the osculating plane, $\widehat{T}$, $\widehat{N}$ and $\widehat{B}$ at $\left(1, \frac{\sqrt{2}}{2}, \frac{1}{3}\right)$.

(g) Find an expression for the curvature, $\kappa(t)$ and the torsion, $\tau(t)$. What do you notice?

10. Consider the parabola: $x = y^2$. In class, we parametrized this curve as $\vec{r}(t) = \langle t^2, t \rangle$ and found:

$$\hat{T}(t) = \left\langle \frac{2t}{\sqrt{4t^2+1}}, \frac{1}{\sqrt{4t^2+1}} \right\rangle, \qquad \hat{N}(t) = \left\langle \frac{1}{\sqrt{4t^2+1}}, -\frac{2t}{\sqrt{4t^2+1}} \right\rangle$$

We also discussed the formula for the curvature of a plane curve $y = f(x)$:

$$\kappa = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}$$

(a) Explain why if $x = g(y)$, the formula for curvature can be found by: $\kappa = \dfrac{|g''(y)|}{(1 + [g'(y)]^2)^{3/2}}$.

NOTE: I'm looking for a sentence (or two!) not a full-blown derivation.

(b) Find the curvature and radius of curvature at $(x, y) = (1, -1)$.

(c) Find $\hat{N}(-1)$ and use your answer to part (b) to help you find the circle of curvature at $(1, -1)$.

Check your answer graphically.