MATH 2700: TEST 01 (100 points.)

NAME: ____

DIRECTIONS: Show all work.

1. (a) A vector, $\vec{v}$ is a mathematical object defined by two quantities:

   i. its ____________________, denoted by $\|\vec{v}\|$, and ...

   ii. its ____________________, denoted by $\widehat{v}$.

   (b) A quick way to check if two nonzero vectors $\vec{v}$ and $\vec{w}$ are orthogonal is to show ______________.

   (c) To construct a vector orthogonal to two nonzero vectors $\vec{v}$ and $\vec{w}$, we can compute ____________.

   (d) Find a vector **parallel** to the line $\vec{L} = \langle 2 - 3t, 6 + t, 5 \rangle$.

   (e) Find a vector **normal** to the plane: $3x + 2y + z = 6$.

   (f) Show the intersection of the cone $z^2 = x^2 + y^2$ and the plane $z = y + 1$ is a parabola.

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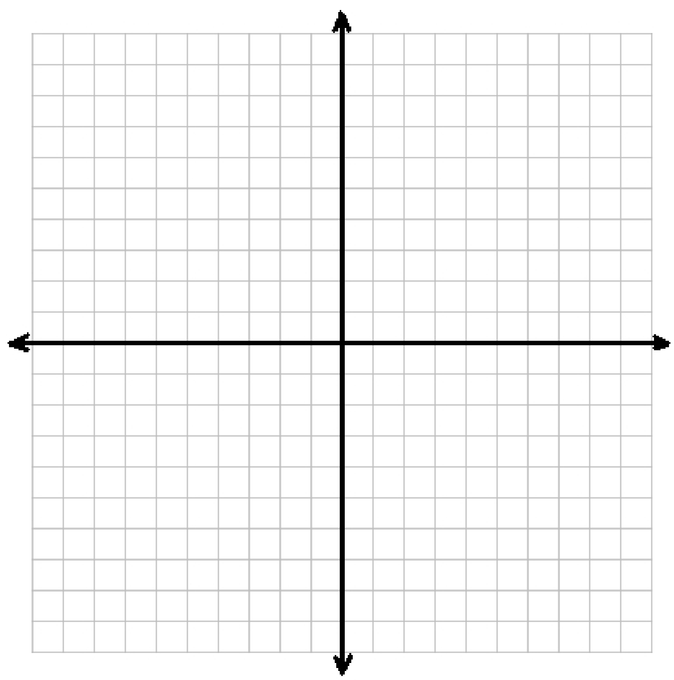
2. An object in the xy-plane positioned at the origin is being acted upon by two forces. One force, $\vec{F_1}$ has a magnitude of 8 N and acts at an angle of 60° with respect to the positive x-axis. The second force, $\vec{F_2}$ is pulling directly left with a force of 4 N.

   (a) Find the component forms of $\vec{F_1}$ and $\vec{F_2}$.

   (b) Find the magnitude and direction of the force required to keep the object from moving, $\vec{v}$.

   Explain your reasoning.

   (c) Sketch $\vec{F_1}$, $\vec{F_2}$, and $\vec{v}$ on the axes below and check your answer graphically.



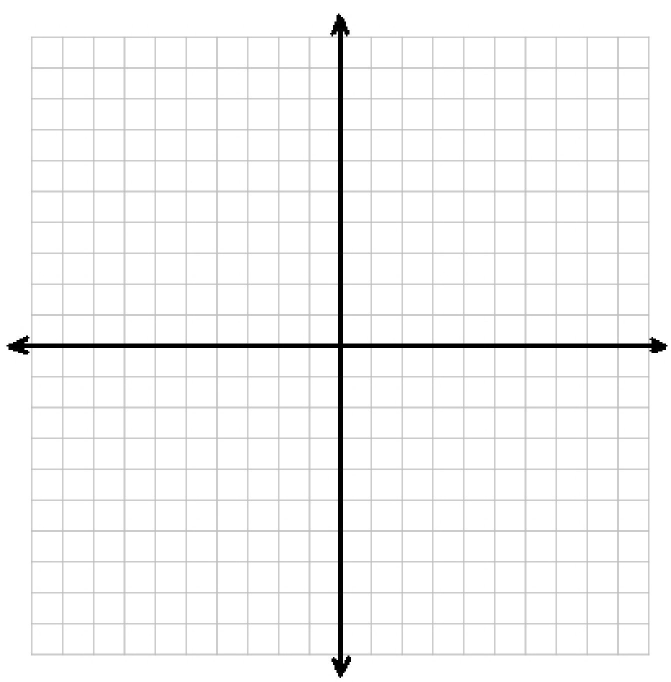
3. Let $\vec{v} =< -5, 10 >$ and $\vec{w} =< 4, -3 >$.

(a) Find and simplify $\vec{p} = \text{proj}_{\vec{w}}\vec{v}$

(b) Find $\vec{q} = \vec{v} - \vec{p}$, and use the dot product to show $\vec{q} \perp \vec{w}$.

(c) On the axes below:

- Graph $\vec{v}$, $\vec{w}$, and $\vec{p}$ in standard position.
- Graph $\vec{q}$ so that the initial point of $\vec{q}$ is the terminal point of $\vec{p}$.
- Geometrically verify that $\vec{v} = \vec{p} + \vec{q}$.



4. Let $P(0, -2, 4)$ and $Q(-2, 1, 10)$.

(a) Find $\overrightarrow{PQ}$, $\| \overrightarrow{PQ} \|$, and $\widehat{PQ}$.

(b) Find the equation of the sphere which contains P and Q as antipodal points.

(c) Find the vector form of the line containing the origin which is parallel to $\overrightarrow{PQ}$.

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(d) Let $R(2, 0, 4)$. Find $\overrightarrow{PR}$ and $\overrightarrow{PQ} \times \overrightarrow{PR}$.

(e) Find the equation of the plane containing P, Q, and R.

Write your answer in the form: $Ax + By + Cz = D$ where $A > 0$.

(f) Find the distance from the origin to the plane containing P, Q, and R.

5. Consider the equation $S : x + y + 2z = 4$.

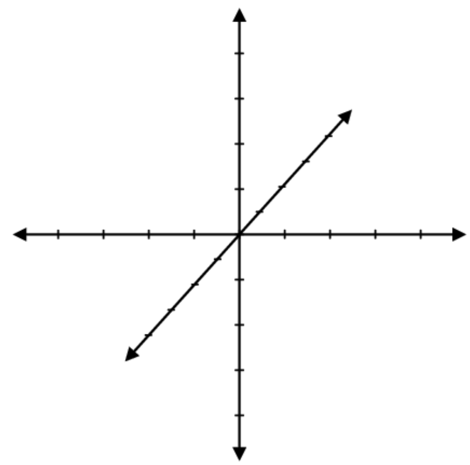
(a) How do you know the graph of S is a plane?

(b) Find the equations for each of the traces and graph S in the first octant. (Label the axis intercepts.)

- xy - trace:

- xz - trace:

- yz - trace:



BONUS: (+5) Find the point on the graph of S which is closest to the origin.

6. Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ be two nonzero vectors. Prove $\vec{v}$ is orthogonal to $\vec{v} \times \vec{w}$.

   HINT: Feel free to use the properties of the triple scalar product ...

7. Explain which properties of the dot product are being used to prove the claim below:

   CLAIM: If $\vec{v}$ and $\vec{w}$ are nonzero vectors, and $\vec{p} = \text{proj}_{\vec{w}}\vec{v}$, then $\vec{q} = \vec{v} - \vec{p}$ is orthogonal to $\vec{w}$.

   PROOF: We show $\vec{w} \cdot \vec{q} = 0$:

$$\begin{aligned}
\vec{w} \cdot \vec{q} &= \vec{w} \cdot (\vec{v} - \vec{p}) && \textbf{REASON: } \vec{q} = \vec{v} - \vec{p} \\
&= \vec{w} \cdot \vec{v} - \vec{w} \cdot \vec{p} && \textbf{REASON: } \underline{\hspace{6cm}} \\
&= \vec{w} \cdot \vec{v} - \vec{w} \cdot \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \, \vec{w} \right) && \textbf{REASON: } \text{definition of } \vec{p} = \text{proj}_{\vec{w}}\vec{v} \\
&= \vec{w} \cdot \vec{v} - \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) (\vec{w} \cdot \vec{w}) && \textbf{REASON: } \underline{\hspace{6cm}} \\
&= \vec{w} \cdot \vec{v} - \frac{\vec{v} \cdot \vec{w}}{\cancel{\vec{w} \cdot \vec{w}}} \, \cancel{\vec{w} \cdot \vec{w}} && \\
&= \vec{v} \cdot \vec{w} - \vec{v} \cdot \vec{w} && \textbf{REASON: } \underline{\hspace{6cm}} \\
&= 0 && \checkmark
\end{aligned}$$

8. Suppose $\widehat{v}$ and $\widehat{w}$ are **orthogonal** and $\vec{u} = a\,\widehat{v} + b\,\widehat{w}$.

(a) Explain why $\|a\,\widehat{v}\| = |a|$ and $\|b\,\widehat{w}\| = |b|$.

(b) Sketch a vector diagram relating $\vec{u}$, $\widehat{v}$, and $\widehat{w}$.

(c) Turn your vector diagram in part (a) into a triangle and explain why $\|\vec{u}\|^2 = a^2 + b^2$.

What famous theorem is this?

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BONUS: Fill in the reasoning below to prove the claim made in 8c using properties of the dot product:

CLAIM: $\|\vec{u}\|^2 = a^2 + b^2$.

PROOF:

$$\begin{aligned}
\|\vec{u}\|^2 &= \vec{u} \cdot \vec{u} \\
&= (a\,\widehat{v} + b\,\widehat{w}) \cdot (a\,\widehat{v} + b\,\widehat{w}) && \textbf{REASON: } \vec{u} = a\,\widehat{v} + b\,\widehat{w} \\
&= (a\widehat{v}) \cdot (a\widehat{v}) + (a\widehat{v}) \cdot (b\widehat{w}) + \ldots \\
&\quad \ldots + (b\widehat{w}) \cdot (a\widehat{v}) + (b\widehat{w}) \cdot (b\widehat{w}) && \textbf{REASON: } \underline{\hspace{6cm}} \\
&= a^2\,(\widehat{v} \cdot \widehat{v}) + ab\,(\widehat{v} \cdot \widehat{w}) + \ldots \\
&\quad \ldots + ba\,(\widehat{w} \cdot \widehat{v}) + b^2\,(\widehat{w} \cdot \widehat{w}) && \textbf{REASON: } \underline{\hspace{6cm}} \\
&= a^2(1) + ab(0) + ba(0) + b^2(1) && \textbf{REASON: } \underline{\hspace{6cm}} \\
&= a^2 + b^2 && \checkmark
\end{aligned}$$

$\overline{+5}$

FORMULAS

- $d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$
- $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$
- $\vec{p} = \dfrac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\, \vec{w} = \dfrac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\, \vec{w}$
- $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$
- $\vec{L}(t) = \overrightarrow{OP} + t\,\vec{v} = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$
- $a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0$
- $d = \dfrac{\left\| \overrightarrow{PQ} \times \vec{v} \right\|}{\|\vec{v}\|}$
- $d = \dfrac{\left| \overrightarrow{PQ} \cdot \vec{n} \right|}{\|\vec{n}\|}$