MATH 2700: TAKE HOME 01 (50 points.)

NAME: ____

DUE: The day of Test 1, at the beginning of class.

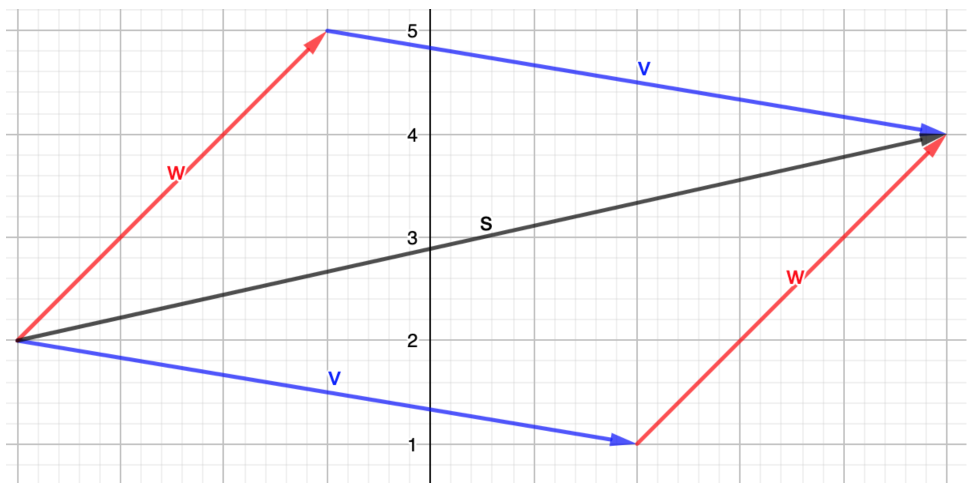
DIRECTIONS: Show all work.

1. A vector, $\vec{v}$ is a mathematical object defined by two quantities:

   (a) its ____________________, denoted by $\|\vec{v}\|$, and ...

   (b) its ____________________, denoted by $\hat{v}$.

2. Consider the vector diagram below:



   Circle all of the vector relationships below which are supported by the diagram:

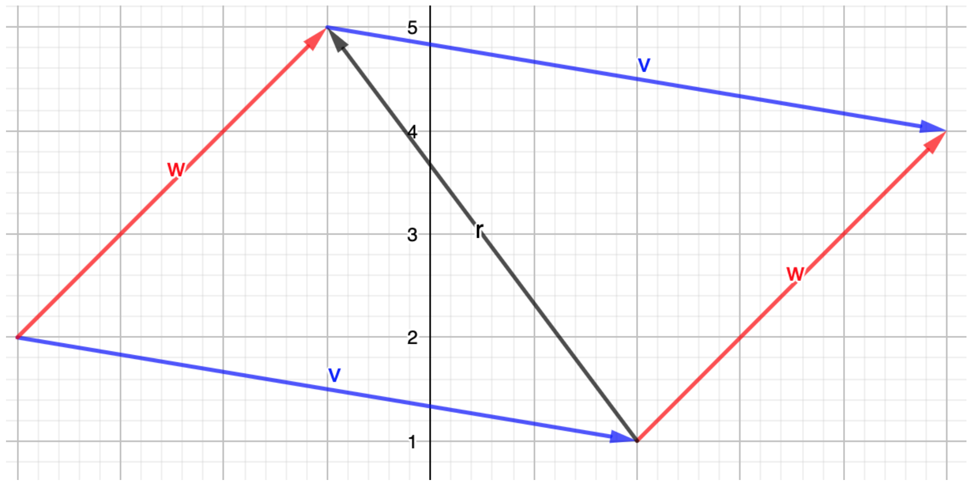
   (a) $\vec{s} = \vec{w} - \vec{v}$

   (b) $\vec{w} = \vec{s} - \vec{v}$

   (c) $\vec{s} = \vec{v} + \vec{w}$

   (d) $\vec{v} = \vec{s} + \vec{w}$

3. Consider the vector diagram below:



Circle all of the vector relationships below which are supported by the diagram:

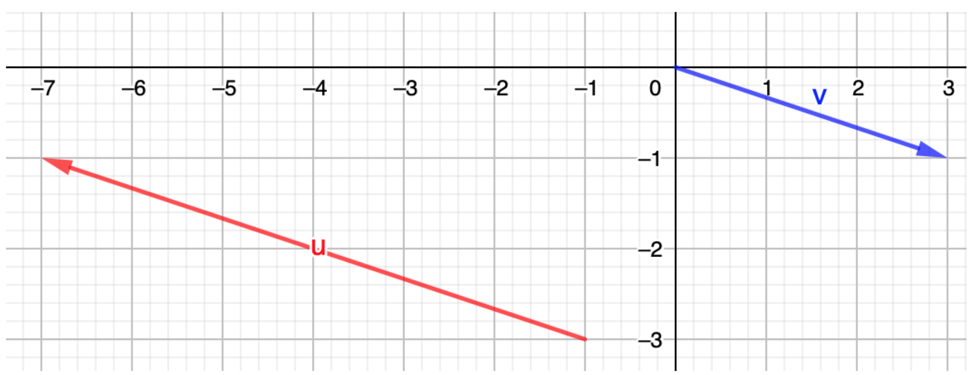
(a) $\vec{w} = \vec{v} + \vec{r}$

(b) $\vec{v} + \vec{r} - \vec{w} = \vec{0}$

(c) $\vec{r} = \vec{v} + \vec{w}$

(d) $\vec{r} = \vec{w} - \vec{v}$

4. Consider the vector diagram below:



Circle all of the vector relationships below which are supported by the diagram:

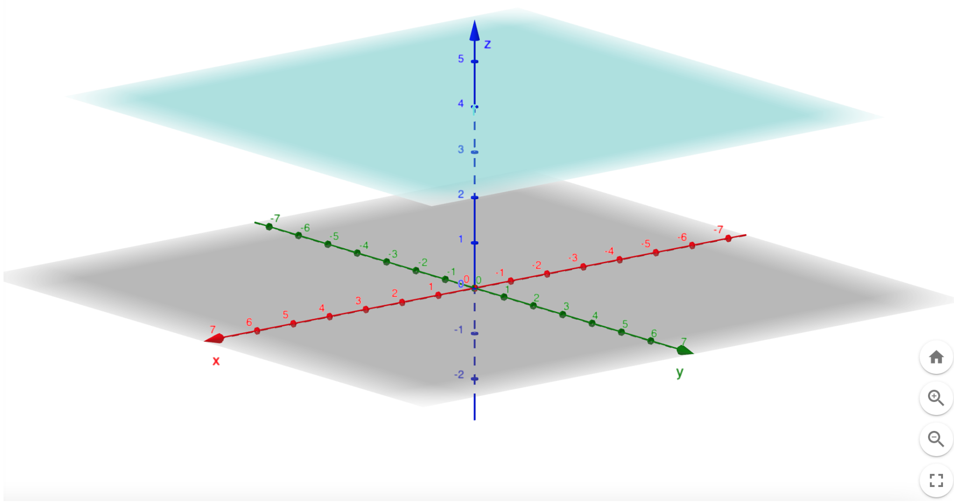
(a) $\vec{u} = 2\vec{v}$

(b) $\vec{u} = -2\vec{v}$

(c) $\|\vec{u}\| = -2\|\vec{v}\|$

(d) $\|\vec{u}\| = 2\|\vec{v}\|$

5. Write an equation for the plane depicted below.



Explain your reasoning and the assumptions you're making.

6. Write an equation for the plane parallel to the yz-plane which contains the point $(1, 2, 3)$.

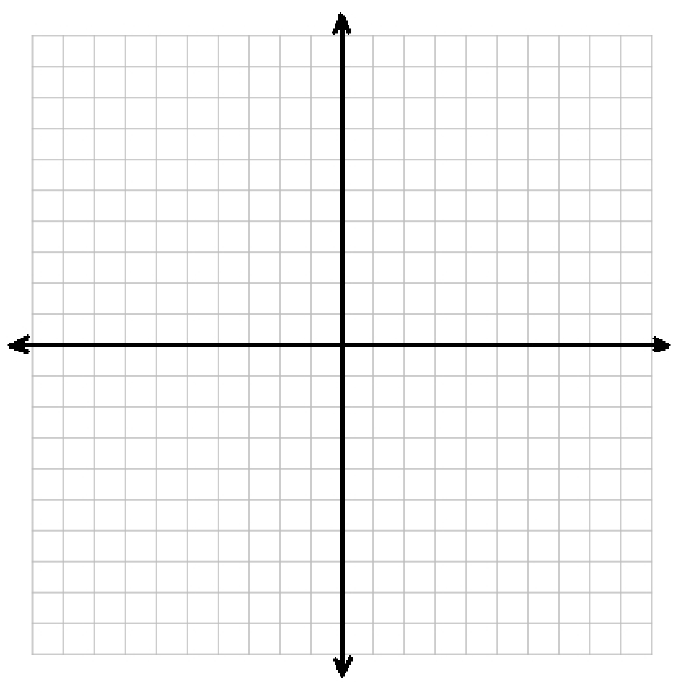
7. An object in the xy-plane positioned at the origin is being acted upon by two forces. One force, $\vec{F_1}$ has a magnitude of 10 lbs. and acts at an angle of 30° with respect to the positive x-axis. The second force, $\vec{F_2}$ acts directly downwards with a force of 5 lbs.

   (a) Find the component forms of $\vec{F_1}$ and $\vec{F_2}$.

   (b) Find the magnitude and direction of the force required to keep the object from moving, $\vec{v}$.

   Explain your reasoning.

   (c) Sketch $\vec{F_1}$, $\vec{F_2}$, and $\vec{v}$ on the axes below and check your answer graphically.



8. Consider the sphere S: $(x-1)^2 + (y+2)^2 + z^2 = 25$.

   (a) Let C be the center of the S. What are the coordinates of C?

   (b) Show the point $P(2, -1, \sqrt{23})$ lies on S.

   (c) What is the radius of S?

   (d) Find the component form of the vector $\overrightarrow{PC}$.

   (e) What should $\| \overrightarrow{PC} \|$ be? Explain. Calculate $\| \overrightarrow{PC} \|$ to confirm your answer.

(f) Look up what it means for two points on a sphere to be **antipodal**.

Find the point Q antipodal to P on S using the point P and the vector $\overrightarrow{PC}$. Explain your reasoning.

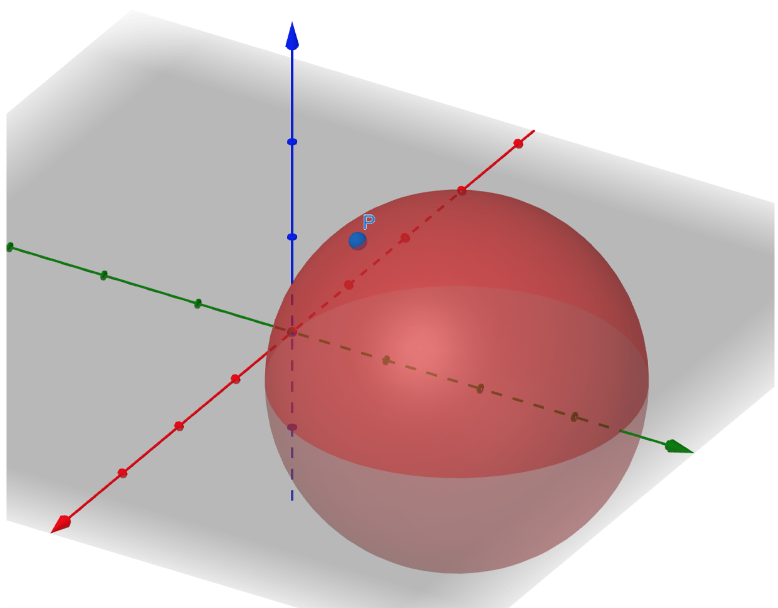
(g) What should $\| \overrightarrow{PQ} \|$ be? Explain. Calculate $\| \overrightarrow{PQ} \|$ to confirm your answer.

(h) Use a graphing utility to graph S, P, C, and Q, and $\overrightarrow{PQ}$ with initial point P to check your answer.

(i) Algebraically show that slicing S with the plane $z = 3$ results in a circle.

9. Find all points of the form $P(a, a, 3a)$ which lie on the sphere: $(x-1)^2 + (y+1)^2 + z^2 = 46$.

10. Find the equation of the sphere which is tangent to the xz-plane and contains both $(1, 2, 3)$ and the origin.



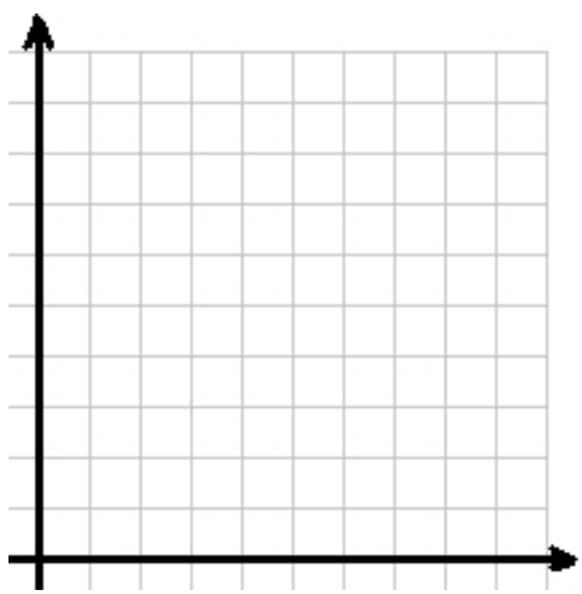
Check your answer using a graphing utility.

11. Let $\vec{v} = < 10, 5 >$ and $\vec{w} = 3\hat{\imath} + 4\hat{\jmath}$.

(a) Find and simplify $\vec{p} = \text{proj}_{\vec{w}}\vec{v}$

(b) Find $\vec{q} = \vec{v} - \vec{p}$, and use the dot product to verify $\vec{q}$ is orthogonal to $\vec{w}$.

(c) Graph $\vec{v}$ and $\vec{w}$ in standard position. Graph $\vec{p}$ in standard position and $\vec{q}$ so that the initial point of $\vec{q}$ is the terminal point of $\vec{p}$ to show geometrically $\vec{v} = \vec{p} + \vec{q}$.



12. Let $\vec{L}$ be the line containing $P(3, 4, 0)$ and $Q(1, 2, -1)$.

(a) Find a vector, parametric, and symmetric description of $\vec{L}$.

- vector form:

- parametric form:

- symmetric form:

(b) Find a vector parallel to $\vec{L}$ and find an equation of the plane through the origin which is normal to $\vec{L}$.

13. (a) Explain which properties of the dot are being used to prove the claim below:

CLAIM: If $\vec{u} \cdot \vec{v} = 0$ and $\vec{u} \cdot \vec{w} = 0$, then $\vec{u} \cdot (\vec{v} + k\,\vec{w}) = 0$ for every real number k.

PROOF: Suppose $\vec{u} \cdot \vec{v} = 0$ and $\vec{u} \cdot \vec{w} = 0$ and k is a real number. Then:

$\vec{u} \cdot (\vec{v} + k\,\vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot (k\,\vec{w})$ **REASON:** ____________________

$= 0 + k\,(\vec{u} \cdot \vec{w})$ **REASON:** ____________________

$= k(0)$

$= 0$ ✓

(b) Use properties of the cross product to prove the following claim:

CLAIM: If $\vec{u} \times \vec{v} = \vec{0}$ and $\vec{u} \times \vec{w} = \vec{0}$, then $\vec{u} \times (\vec{v} + k\,\vec{w}) = \vec{0}$ for every real number k.

PROOF: Suppose $\vec{u} \times \vec{v} = \vec{0}$ and $\vec{u} \times \vec{w} = \vec{0}$ and k is a real number. Then:

$\vec{u} \times (\vec{v} + k\,\vec{w}) =$

14. Consider the line: $\vec{L}(t) = \langle 2t - 1, t + 2, 4 - 2t \rangle$.

Let's find the point P on $\vec{L}$ closest to the origin, $O(0, 0, 0)$, using three different[1] methods.

(a) i. Calculate the distance from O to $\vec{L}$ using the vector formula we developed in class.

How does your answer prove that $\vec{L}$ does not contain O?

[1] yet equally exciting

ii. Remember that every point P on $\vec{L}$ has coordinate of the form: $P(2t-1, t+2, 4-2t)$.

Find an expression for the distance between O and P, $d(t)$.

Set $d(t)$ equal to your answer to part (i) and solve for t.

iii. Find the point on $\vec{L}$ closest to O.

(b) Now let's try a second method which Calculus applied to $d(t)$ from part (a).

i. Find $d'(t)$.

ii. Make a Sign Diagram for $d'(t)$ and find the value of t which minimizes $d(t)$.

iii. Find the point on $\vec{L}$ closest to O.

(c) Now let's try another[2] method.

i. Find a vector, $\vec{v}$ which is parallel to $\vec{L}$.

ii. Find the point P on $\vec{L}$ which is closest to O by solving $\overrightarrow{OP} \cdot \vec{v} = 0$.

iii. Check your answer by showing the distance between O and $\vec{L}$ matches part (a) number (i).

iv. Draw a picture (or two!) explaining what is happening here geometrically.

[2]the most elegant?

15. Let $P(1,-2,3)$, $Q(0,1,4)$, and $R(-2,0,7)$ and let W be the plane containing P, Q, and R.

(a) Find $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PQ} \times \overrightarrow{PR}$.

(b) Find the equation of W and explain why the origin, $O(0,0,0)$ is not on W.

(c) Calculate the distance between O and W using the vector formula we developed in class.

(d) Suppose the point $S(a, b, c)$ is the point on W which is closest to the origin. Let's find S.

NOTE: In two chapters or so, we'll revisit this problem with more powerful mathematical tools!

i. Use the fact that $S(a, b, c)$ is on W to write an equation involving a, b, and c.

ii. Explain why the vector $\overrightarrow{OS} = \langle a, b, c \rangle$ must be **parallel** to the vector $\overrightarrow{PQ} \times \overrightarrow{PR}$.

Hence, $\overrightarrow{OS} = k \left(\overrightarrow{PQ} \times \overrightarrow{PR} \right)$ for some real number k.

iii. Equate components of the vectors in the equation $\overrightarrow{OS} = k \left(\overrightarrow{PQ} \times \overrightarrow{PR} \right)$.

You should now have expressions for each of a, b, and c in terms of k.

iv. Substitute the expressions for a, b, and c in terms of k into the equation you have from the first part of this problem which relates a, b, and c.

Solve for k, which, in turn, lets you solve for a, b, and c.

v. Show that $S(a, b, c)$ is on W and calculate the distance between O and S to check your answer.

16. Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$, and $\vec{w} = \langle w_1, w_2, w_3 \rangle$.

(a) Find and simplify an expression for $\vec{v} \times \vec{w}$ in terms of the v_i's and w_i's.

(b) Find an expression for $\vec{u} \cdot (\vec{v} \times \vec{w})$ in terms of the u_i's, v_i's and w_i's

(c) Use your answer to part (b) to show: $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$.

17. Suppose $\widehat{u}$ and $\widehat{v}$ are orthogonal unit vectors. Let $\vec{w} = \widehat{u} \times \widehat{v}$.

(a) Explain why we know $\vec{w}$ is orthogonal to both $\widehat{u}$ and $\widehat{v}$.

(b) Use the formula $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \, \sin(\theta)$ to show $\vec{w}$ is a **unit** vector so we may write $\vec{w} = \widehat{w}$.

NOTE: Here, the relationships between $\widehat{u}$, $\widehat{v}$, and $\widehat{w}$ resemble those between $\widehat{i}$, $\widehat{j}$, and $\widehat{k}$.

18. **SCALAR, VECTOR, OR NONSENSE:** Assume $\vec{u}$, $\vec{v}$, and $\vec{w}$ are nonzero 3D vectors.

Label each quantity below as resulting in a '**s**calar,' '**v**ector,' or state the quantity is '**n**onsense.'

(a) $\vec{v} \cdot \vec{w}$

(b) $\vec{v} \times \vec{w}$

(c) $\vec{u} \cdot (\vec{v} \cdot \vec{w})$

(d) $\vec{u} \times (\vec{v} \times \vec{w})$

(e) $\vec{u} \cdot (\vec{v} \times \vec{w})$

(f) $\vec{u} \times (\vec{v} \cdot \vec{w})$

(g) $\dfrac{\vec{v}}{\|\vec{w}\|} \, \vec{w}$

(h) $\dfrac{\vec{v}}{\|\vec{w}\|} \cdot \vec{w}$

(i) $\dfrac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|} \, \vec{w}$

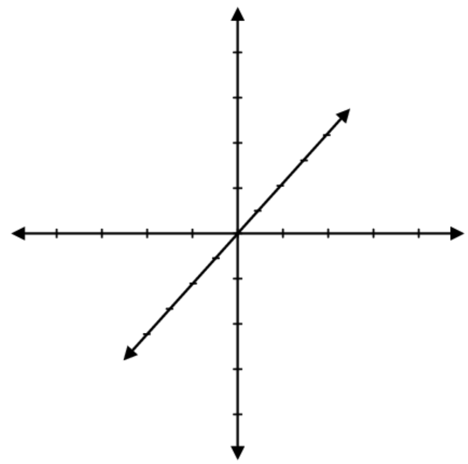
(j) $\dfrac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|} \cdot \vec{w}$

19. Consider the equation $S : 3x + 2y + z = 6$.

(a) How do you know the graph of S is a plane?

(b) Find the equations for each of the traces and graph S in the first octant. (Label the axis intercepts.)

- xy - trace:
- xz - trace:
- yz - trace:



(c) i. Show the point $P\left(\frac{1}{3}, \frac{1}{2}, 4\right)$ is on the graph of S.

ii. Find a vector form of the equation of the line through P which is normal to the graph of S.