MATH 2600: TAKE HOME 04 (20 POINTS)

DUE THE DAY OF TEST 4 AT THE BEGINNING OF CLASS

NAME: ____

DIRECTIONS: Make sure your work is neat and complete and uses the techniques demonstrated in class.

SECTION 6.2 PRACTICE PROBLEMS

1. Let R be the region in the first quadrant bounded by $y = 3\sqrt{x}$, $y = 4 - x$ and the x-axis.

   Set-up, **but do not evaluate**, an integral or sum of integrals which would determine the **area** of R:

   (a) by chopping-up the x-axis

   (b) by chopping up the y-axis

2. In economics, a **Lorenz Curve**, $L(x)$ gives the percentage of the total national income earned by the bottom x percent of wage earners, ranked from lowest income to highest income. So, for example, $L(0.2)$ is the percentage of money income earned by the lowest 20% of wage earners; the number $L(0.4)$ is the percentage of the money income earned by the bottom 40% of wage earners; $L(0.6)$ is the total income share of the bottom 60% of wage earners, and so on. By definition, then, $L(0) = 0$ and $L(1) = 1$.

   If wealth is divided equally among wage earners, the bottom x% of wage earner would earn x% of the national income. That is, $L(x) = x$, In reality, $L(x) < x$ so we define the Gini Index to measure the gap:

$$G = 2\int_0^1 (x - L(x))\,dx$$

   Find the Gini Index of the following scenarios.

   (a) $L(x) = x$ (equal division of income.)

   (b) $L(x) = 0$ if $0 \le x < 1$, $L(1) = 1$ (one person has it all.)

   (c) $L(x) = 0.982865\,x^{2.83598}$ (the Lorenz Curve as fitted from the 2020 US Census, Table A-3.)

   Feel free to use desmos for the heavy lifting here.

SECTION 6.3 / 6.4 PRACTICE PROBLEMS

1. Let R be the region between the graphs of $y = \sin(x)$ and $y = \cos(x)$ for $0 \leq x \leq \frac{\pi}{4}$.

   Find the volume of the solid obtained by revolving R about the x-axis.

   HINT: $\cos^2(x) - \sin^2(x) = \cos(2x)$ ...

2. Let $R = \left\{ (x, y) : 0 \leq x \leq \dfrac{\pi}{4}, \sin(x) \leq y \leq \cos(x) \right\}$.

Find the volume of the solid obtained by revolving R about the line $x = \dfrac{\pi}{2}$.

HINT: Parts!

ARC LENGTH PRACTICE PROBLEMS (SECTION 6.5, 12.1, 12.3)

1. Let $f(x) = \ln(x) - \dfrac{x^2}{8}$.

   (a) Find and simplify the arc length differential, ds.

   (b) Find the arc length of the graph of $y = f(x)$ from $x = 1$ to $x = e$ using your answer to part (a).

2. Let C be the curve traced out by the parametric equations:

$$\begin{cases} x = 3\cos(2t) - 1 \\ y = 3\sin(2t) + 2 \end{cases}, \quad 0 \le t \le \pi$$

(a) Find and simplify the arc length differential, ds.

(b) Find the arc length of C using you answer to part (a).

(c) Prove that C is a circle and check your answer to part (b) using a formula from geometry.

3. Find the arc length of the curve C traced out by the parametric equations:

$$\begin{cases} x(t) &= 4\cos^3(t) \\ y(t) &= 4\sin^3(t) \end{cases}, \quad 0 \leq t \leq 2\pi.$$

NOTE: This is the same curve from Take Home 03...

4. Find the arc length of the Archimedian spiral described in polar coordinates by $r = a\,\theta$, $0 \le \theta \le 1$.

Assume $a > 0$ is a constant.

SECTION 6.7 PRACTICE PROBLEMS

1. (a) Show the **work done** stretching a spring with spring constant k a total 'b' units from the spring's **equilibrium position** is given by:

$$W = \frac{1}{2}\, k\, b^2.$$

   (b) Suppose the **work done** stretching a spring 3 m from its equilibrium position is 12 J. Find the work done stretching the spring an additional 5 m.

2. Set-up, **but do not evaluate**, an integral which computes the work done against gravity in launching a 100 kg satellite vertically into an orbit of 500 km above Mercury.

   Recall, the force of gravity, F, between two objects is given by:

$$F = G\frac{Mm}{r^2},$$

   where M and m are the masses of the two objects and r is the distance between them.

   Assume the mass of Mercury is 3.30×10^{23} kg, all of which is concentrated at its center. Other useful information: the radius of Mercury is 2440 km, and the gravitational constant, $G = 6.67 \times 10^{-11}$ Nm^2/kg^2.

3. The base of an above ground rectangular swimming pool measures 16 ft. by 32 ft. and and is 4 ft. deep. Suppose the pool is partially filled so the water in the pool is 3 ft. deep. Set up and evaluate an integral to compute the work done pumping all of the water in the pool to the top of the pool. Assume the weight-density of water is 62.5 $\frac{\text{lbs}}{\text{ft}^3}$. For a bonus, find the work done using a center of mass calculation!

   NOTE: Include a detailed diagram which explains the setup of your integral.