MATH 2600: TAKE HOME 03 (20 POINTS)

DUE THE DAY OF TEST 3 AT THE BEGINNING OF CLASS

NAME: ______________________________

DIRECTIONS: Make sure your work is neat and complete and uses the techniques demonstrated in class.

SECTION 11.1 PRACTICE PROBLEMS

1. Suppose $f(2) = 1$, $f'(2) = -2$, $f''(2) = 3$, and $f'''(2) = -4$.

   (a) Find the third degree Taylor polynomial for f centered at $a = 2$, $p_3(x)$.

   (b) Use $p_3(2.1)$ to approximate $f(2.1)$.

   (c) If $|f^{(4)}(x)| \leq 12$ on the interval $[2, 2.1]$, estimate the error in using $p_3(2.1)$ to approximate $f(2.1)$.

2. (a) Find the quadratic approximation to $f(x) = (1+x)^n$ centered at $x = 0$.

(b) Use part (a) to find the quadratic approximation for $g(x) = (1-x)^{-n}$ centered at 0.

HINT: $g(x) = (1-x)^{-n} = (1+(-x))^{(-n)}$ so you use part (a) and make the appropriate substitutions.

(c) Use the results from parts (a) and (b) to show if $x \approx 0$:

$$(1+x)^n - (1-x)^{-n} \approx -nx^2$$

SECTION 11.2 PRACTICE PROBLEMS

1. Consider the power series: $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k (x-1)^{3k+1}}{2^{2k}}$.

   (a) Prove the power series is geometric power series by showing the ratio $\dfrac{a_{k+1}}{a_k}$ is independent of k.

   (b) Determine the interval of convergence and list the center and radius of convergence.

(c) Use the Geometric Series Sum formula to find the rational function to which the power series converges.

Check your answer using desmos.

2. Let $f(x) = \sum_{k=0}^{\infty} c_k (x-3)^k$ where c_k are real numbers. Suppose $f(5)$ converges.

(a) What is the center of this series?

(b) What, if anything, can be said about $f(4)$? Explain.

(c) What, if anything, can be said about $f(2)$? Explain.

(d) What, if anything, can be said about $f(1)$? Explain.

3. Find the interval of convergence for: $\sum_{k=0}^{\infty} \frac{(-1)^k (x-3)^k}{(k+1)\, 2^{2k+1}}$

4. Suppose $f(x)$ is a power series with interval of convergence $[-2, 4)$.

(a) What is the center of this series?

(b) What is the radius of convergence of this series?

(c) What are the possibilities for the interval of convergence for $f'(x)$?

(d) What are the possibilities for the interval of convergence for $F(x) = \int_1^x f(t)\, dt$?

5. Consider the function: $f(x) = \dfrac{18}{8 - x^2 + 2x}$.

(a) Complete the square in the denominator and find a power series representation for f centered at $a = 1$.

(b) Find the interval of convergence of the power series you found in part (a).

(c) Check your answer to part (a) by using the geometric series sum formula and simplifying your answer.

6. **EXPLORATION:**

Let $y = 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(2\cdot 3)(5\cdot 6)\cdots(3k-1)(3k)} = 1 + \frac{x^3}{(2)(3)} + \frac{x^6}{(2)(3)(5)(6)} + \frac{x^9}{(2)(3)(5)(6)(8)(9)} + \ldots$

(a) Show $y'' = x + \sum_{k=2}^{\infty} \frac{x^{3k-2}}{(2\cdot 3)(5\cdot 6)\cdots(3k-4)(3k-3)}$. **HINT:** When in doubt ...

(b) Shift the index in the sum: $\sum_{k=2}^{\infty} \frac{x^{3k-2}}{(2\cdot 3)(5\cdot 6)\cdots(3k-4)(3k-3)}$ from k to $n = k-1$ and show:

$$y'' = x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{(2\cdot 3)(5\cdot 6)\cdots(3n-1)(3n)}$$

(c) Use parts (a) and (b) to show y satisfies the differential equation: $y'' - xy = 0$.

NOTE: The equation: $y'' - xy = 0$ is known as Airy's Equation is a nice topic for a project!

SECTION 11.3 PRACTICE PROBLEMS

1. In statistics, probabilities of certain events are computed by finding the area under continuous functions called **probability density functions** or 'p.d.f.'s. For a population with mean μ and standard deviation σ, the p.d.f. for the so-called **normal distribution** is given by:

$$f(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\dfrac{(x-\mu)^2}{2\sigma^2}}$$

If $\mu = 0$ and $\sigma = 1$, this distribution reduces to: $f(x) = \dfrac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$.

NOTE: Graphing this on desmos reveals it to be one of the so-called 'bell' curves.

(a) Use desmos to graph $f(x) = \dfrac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$ and sketch below the area associated with $\displaystyle\int_{-2}^{2} f(x)\,dx$.

(b) Prove f is an **even** function[1] and use that to show $\displaystyle\int_{-2}^{2} f(x)\,dx = 2\int_{0}^{2} f(x)\,dx$

[1]Recall this means $f(-x) = f(x)$

(c) Use the Maclaurin series for $F(x) = e^x$ to find and simplify a series representation for $f(x)$.

(d) Using part (c), find a series for: $2\int_0^2 f(x)\,dx$.

(e) Approximate $2\int_0^2 f(x)\,dx$ by adding the first 10 terms of your answer to part (d).

RECALL: If your counter k starts at $k = 0$, then what value of k produces the 10th term?

(f) Use the AST Remainder Theorem to estimate the error in your approximation in part (e).

(g) Research the '68 - 95 - 99' rule. What does this have to do with your answer to part (e)?

SECTION 12.1 PRACTICE PROBLEMS

1. The Curve C below is known as the Folium of Descartes.

$$\begin{cases} x(t) &= \dfrac{3t}{t^3+1} \\ y(t) &= \dfrac{3t^2}{t^3+1} \end{cases}, \quad t \neq -1$$

(a) Find the following limits.

- $\lim\limits_{t\to-\infty} x(t)$
- $\lim\limits_{t\to-\infty} y(t)$
- $\lim\limits_{t\to-1^-} x(t)$
- $\lim\limits_{t\to-1^-} y(t)$
- $\lim\limits_{t\to-1^+} x(t)$
- $\lim\limits_{t\to-1^+} y(t)$
- $\lim\limits_{t\to\infty} x(t)$
- $\lim\limits_{t\to\infty} y(t)$

(b) Use the limits in part (a) along with desmos to graph C and sketch the curve, complete with orientation.

(c) Find $\lim_{t \to -1} [x(t) + y(t) + 1]$ and interpret what this means graphically.

HINT: Graph $x + y + 1 = 0$ along with the Folium on desmos.

(d) Find the equation of the tangent line to C at when $t = 1$. Check your answer using desmos.

2. Consider the curve, C, described below:

$$\begin{cases} x(t) &= 4\cos^3(t) \\ y(t) &= 4\sin^3(t) \end{cases}, \quad 0 \le t \le 2\pi.$$

(a) Find and simplify expressions for $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$, and $\dfrac{dy}{dx}$.

List the open intervals of t over which C is increasing and decreasing.

List the points (x, y) where t is not smooth.

(b) Find and simplify an expression for $\dfrac{d^2y}{dx^2}$.

List the open intervals of t over which the C is concave up / down.

(c) Use the information you've gathered to sketch the graph of this curve. Label the key points.

Check your answer using desmos.

(d) **EXPLORATION:** Find the area enclosed by C. Use symmetry and Wallis' Formula: if n is even:

$$\int_0^{\frac{\pi}{2}} \sin^n(\theta)\,d\theta = \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} \cdot \frac{\pi}{2}$$

For example:

$$\int_0^{\frac{\pi}{2}} \sin^2(\theta)\,d\theta = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}, \quad \int_0^{\frac{\pi}{2}} \sin^4(\theta)\,d\theta = \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2} = \frac{3\pi}{16}, \quad \int_0^{\frac{\pi}{2}} \sin^6(\theta)\,d\theta = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{\pi}{2} = \frac{15\pi}{96}$$

SECTION 12.2 and 12.3 PRACTICE PROBLEMS

1. Consider the curve in the plane described be the equation: $x^3 + y^3 = (x^2 + y^2)^2$.

   (a) Convert this equation to polar. Solve for r.

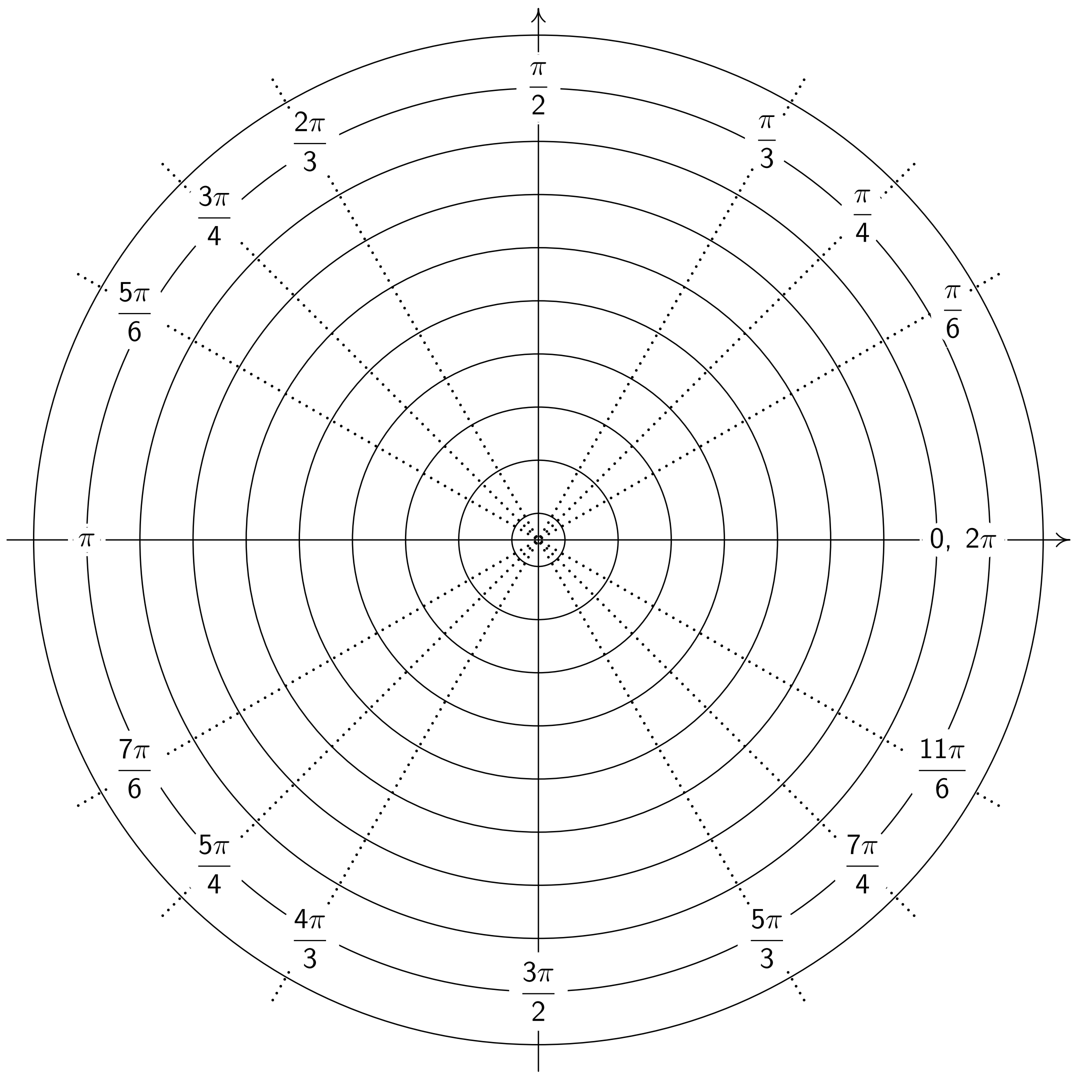
   (b) Find the equation of the tangent line at the origin.[2] Check your answer using desmos.

[2] This is the famous 'bean curve'. We finally have the tools to analytically get the tangent line at $(0, 0)$.

2. (a) Sketch a detailed graph of the polar equation $r = 1 + 2\sin\left(\dfrac{\theta}{2}\right)$ in the xy-plane.

Note the orientation and label the tangents at the pole and any key points.

Check your answer using Desmos. **HINT:** Plot in increments of $\dfrac{\pi}{3}$ ...



(b) **SET-UP** but do **NOT** evaluate an integral or a sum of integrals which would compute the following.

Use symmetry and desmos to help; include a sketch to explain your reasoning.

i. The area enclosed by the outer loop.

ii. The area enclosed by one of the inner loops.

iii. The area common to both inner loops.