MATH 2600: TAKE HOME 02 (20 POINTS)

NAME: ______________________________

DIRECTIONS: Make sure your work is neat and complete and uses the techniques demonstrated in class.

SECTION 7.6 PRACTICE PROBLEMS

1. (a) What two indeterminate forms does L'Hopital's Rule apply to?

   (b) Consider the limit: $\lim_{x \to 0} \dfrac{1 + 3x - e^{3x}}{\cos(2x) - 1}$

      i. Identify which indeterminate form is present in the limit.

      ii. Evaluate the limit with the help of L'Hopital's Rule.

2. Consider the limit: $\lim_{x\to\infty} \sqrt{x}\, e^{-x}$

(a) Identify which indeterminate form is present in the limit.

(b) Evaluate the limit with the help of L'Hopital's Rule.

3. (a) Consider the limit: $\lim_{x\to\infty}\left[\left(\frac{x-1}{x}\right)^{x}\right]$

   i. What indeterminate form is present in this limit?

   ii. Use L'Hopital's Rule to determine this limit.

(b) Use part (a) to help you find: $\lim_{x\to\infty}\left[1-\left(\frac{x-1}{x}\right)^{x}\right]$.

NOTE: This limit was used by Carl's friend, Kyle, to assist him in 'shiny hunting' in Pokemon Violet...

4. Assume $a, b > 0$ and consider the limit: $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{bx}$.

(a) What indeterminate form is present in this limit?

(b) Let $y = \left(1 + \frac{a}{x}\right)^{bx}$. Use a log property to show: $\ln(y) = bx \ln\left(1 + \frac{a}{x}\right)$.

(c) Consider the limit: $\lim_{x \to \infty} \ln(y) = \lim_{x \to \infty} bx \ln\left(1 + \frac{a}{x}\right)$.

i. What indeterminate form is present in this limit?

ii. Rewrite as needed and use L'Hopital's Rule to help you determine the limit.

(d) Use your answer to part (c) to help you determine $\lim_{x\to\infty} y = \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$.

RECALL: Since the exponential function is continuous: $\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln(y)} = e^{\lim_{x\to\infty}\ln(y)}$.

(e) Use your answer to part (d) to show: $\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = Pe^{rt}$.

NOTE: This shows the the formula for **continuously compounded** interest is the limit of the formula for interest compounded n times per year as $n \to \infty$.

SECTION 8.9 PRACTICE PROBLEMS

1. For the integral $\int_0^1 \sqrt{x}\, \ln(x)\, dx$:

   (a) Explain why the integral is 'improper' and rewrite the integral as the limit of a 'proper' integral.

   (b) Evaluate the 'proper' integral and take the limit.[1]

   NOTE: Be sure to indicate where L'Hopital's Rule is needed when evaluating the limit ...

   (c) Explain whether the integral converges or diverges.

[1] Feel free to re-use some of your work from Take Home 01.

2. For the integral $\displaystyle\int_0^{\infty} x\, e^{-2x}\, dx$:

(a) Explain why the integral is 'improper' and rewrite the integral as the limit of a 'proper' integral.

(b) Evaluate the 'proper' integral and take the limit.

NOTE: Be sure to indicate where L'Hopital's Rule is needed when evaluating the limit ...

(c) Explain whether the integral converges or diverges.

3. In Take Home 01, you showed that if s and ω are positive constants, then:

$$\int e^{-st} \cos(\omega t)\, dt = \frac{e^{-st}\left[\omega \sin(\omega t) - s \cos(\omega t)\right]}{s^2 + \omega^2} + C$$

In this Take Home, we wish to find the value of $\displaystyle\int_0^{\infty} e^{-st} \cos(\omega t)\, dt$.

(a) Use the formula you derived in Take Home 01 to determine: $\displaystyle\int_0^{b} e^{-st} \cos(\omega t)\, dt$.

(b) Explain why if $s > 0$, $\displaystyle\lim_{b\to\infty} e^{-sb} = 0$.

(c) Use the Squeeze Theorem to show if $s > 0$, then $\lim_{b\to\infty} e^{-sb}\sin(\omega b) = 0$ and $\lim_{b\to\infty} e^{-sb}\cos(\omega b) = 0$.

HINT: Remember, for all angles θ, $-1 \leq \sin(\theta) \leq 1$ and $-1 \leq \cos(\theta) \leq 1$.

(d) Put your answers from parts (a) and (c) together to find $\int_0^\infty e^{-st}\cos(\omega t)\,dt$.

SECTION 10.1 / 10.2 PRACTICE PROBLEMS

1. Find the limit of the following sequences by passing to a continuous variable. Show all your work.

   (a) $\lim_{n\to\infty}\left[\tan^{-1}(3n)+\cos\left(\frac{2}{n}\right)\right]$.

   (b) $\lim_{k\to\infty}\frac{\sqrt{25k^2+k-2}}{4-3k}$.

2. For each of the limits below:

- Explain why passing to a continuous variable does not work to find the limit.
- Use the Squeeze Theorem to determine each of the limits. Show all work and explain your reasoning.

(a) $\lim_{n\to\infty} \frac{(-1)^n}{5n-1}$.

(b) $\lim_{k\to\infty} \frac{\sin(3k)}{2k+1}$.

3. Consider the sequence $a_n = \dfrac{n!}{n^n}$, $n \geq 1$.

(a) Numerically investigate this sequence. What does the limit appear to be?

(b) It turns out that $a_n \leq \dfrac{1}{n}$ for $n \geq 1$. Use this fact to help you prove what the limit of the sequence is.

(c) **CHALLENGE!** Algebraically prove $a_n \leq \dfrac{1}{n}$.

4. **EXPLORATION:** We've seen sequences that model linear growth (arithmetic sequences) and exponential growth (geometric sequences). We can also model logistic growth using sequences as well! You may remember logistic growth from Calculus 1 or College Algebra. Logistic growth incorporates exponential growth but also the notion of a population limit or 'carrying capacity.' The key principle is that the growth rate is jointly proportional to both the population and the amount of space left for the population to grow.

   Formally, If P_0 represents the initial population of a species which grows at a rate r within an environment of carrying capacity K, then logistic growth is modeled by:

$$P_n = P_{n-1} + r\left(1 - \frac{P_{n-1}}{K}\right)P_{n-1}, \quad n \geq 1$$

   (a) Suppose a lake is currently home to 150 specimens of the invasive Ippizuti fish. If the Ippizuti fish population grows at 60 % per year and the lake is estimated to sustain a maximum of 1500 Ippizuti fish, find a recursive sequence P_n which models the population of Ippizuti fish in the lake in n years.

   (b) Use a graphing utility or a spreadsheet to find the population of Ippizuti fish for the first 15 years. Plot the data and sketch the shape below or attach a screenshot. Do you see how the population of the Ippizuti fish grows more rapidly early on then growth rate tapers off as time goes by? At what year did the growth of the Ippizuti fish population begin to grow less rapidly? Thinking back to curve sketching from Calculus 1, if we connected the data points with a smooth curve, what is the name of the point at which the population of Ippizuti fish begins to grow less rapidly than before?

SECTION 10.3 PRACTICE PROBLEMS

1. Consider the series: $\displaystyle\sum_{k=2}^{\infty} \frac{6}{k^2 - k}$.

   (a) Find an explicit formula for the nth partial sum, S_n.

   (b) Find $\displaystyle\lim_{n\to\infty} S_n$ to determine if the series converges or diverges.

2. The toy Super Happy Fun Ball has the property that it rebounds to $\frac{5}{6}$ of its height after each bounce. Assuming Super Happy Fun Ball is dropped from an initial height of 2 meters and bounces only vertically forever, find the total distance it will travel.

   HINT: Sketch the situation before diving into a geometric series ...

3. Consider the series: $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(x-1)^k}{2^{3k}}$.

(a) Prove the series is geometric and find the common ratio, r.

(b) Determine all values of x for which the series converges. What function does the series converge to?

(c) Use desmos to graphically check your answer to part (b).

SECTION 10.4 PRACTICE PROBLEMS

1. Circle the most correct answer below.

   (a) If $\lim_{n\to\infty} \sum_{k=1}^{n} a_k = 117$, then the series $\sum_{k=1}^{\infty} a_k$:

   - definitely converges.
   - definitely diverges.
   - may or may not converge.

   (b) If $\lim_{n\to\infty} \sum_{k=1}^{n} a_k = -\infty$, then the series $\sum_{k=1}^{\infty} a_k$:

   - definitely converges.
   - definitely diverges.
   - may or may not converge.

   (c) If $\lim_{k\to\infty} a_k = \frac{1}{42}$, then the series $\sum_{k=1}^{\infty} a_k$:

   - definitely converges.
   - definitely diverges.
   - may or may not converge.

   (d) If $\lim_{k\to\infty} a_k = 0$, then the series $\sum_{k=1}^{\infty} a_k$:

   - definitely converges.
   - definitely diverges.
   - may or may not converge.

   (e) If $\lim_{k\to\infty} a_k$ does not exist, then the series $\sum_{k=1}^{\infty} a_k$:

   - definitely converges.
   - definitely diverges.
   - may or may not converge.

2. Determine whether the following series converge or diverge. Be sure to explain your reasoning!

(a) $\displaystyle\sum_{m=1}^{\infty} \cos\left(\frac{\pi}{m}\right)$

(b) $\displaystyle\sum_{m=1}^{\infty} \cos(\pi m)$

(c) $\displaystyle\sum_{k=3}^{\infty} \frac{5k+2}{3-7k}$

(d) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n 7^n}{3^{2n+1}}$

3. Consider the series: $\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right)$.

(a) Explain why the Divergence Test fails for this series.

(b) Find an explicit formula for the nth partial sum, S_n.

HINT: Recall: $\ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B)$

(c) Find $\lim_{n\to\infty} S_n$ to determine if the series converges or diverges.

4. (a) Determine if the following integral converges or diverges: $\displaystyle\int_2^{\infty} \frac{1}{x\,(\ln(x))^3}\,dx$

HINT: Let $u = \ln(x)$ and use what we know about integrals of the form $\displaystyle\int_a^{\infty} \frac{1}{u^p}\,du$.

(b) Use part (a) to help you determine if the following series converges or diverges: $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\,(\ln(k))^3}$.

5. Recall the Integral Test Remainder Theorem: If $\sum_{k=1}^{\infty} a_k$ converges by the integral test, and we define:

$$S_n = \sum_{k=1}^{n} a_k \qquad \text{and} \qquad R_n = \sum_{k=n+1}^{\infty} a_k,$$

Then: $0 \leq R_n \leq \int_n^{\infty} f(x)\,dx$.

(a) Since $\sum_{k=1}^{\infty} \frac{1}{k\sqrt[3]{k}}$ is a p-series, it can be shown to converge using the integral test.

Use the Integral Test Remainder Theorem to find the smallest value of n so that $R_n \leq 0.005$.

(b) Use your 'n' from part (a) to find the sum of $\sum_{k=1}^{\infty} \frac{1}{k\sqrt[3]{k}}$ to within an error of 0.005 the partial sum S_n.

NOTE: Feel free to use Desmos or a graphing calculator to do the heavy lifting here!

SECTION 10.5 PRACTICE PROBLEMS

1. Circle the most correct answer below.

   (a) If $0 < a_k < \frac{1}{\sqrt{k}}$, then the series $\sum_{k=1}^{\infty} a_k$:

   - definitely converges.
   - definitely diverges.
   - may or may not converge.

   (b) If $0 < a_k < \frac{1}{k\sqrt{k}}$, then the series $\sum_{k=1}^{\infty} a_k$:

   - definitely converges.
   - definitely diverges.
   - may or may not converge.

   (c) If $a_k > \frac{1}{\sqrt{k}}$, then the series $\sum_{k=1}^{\infty} a_k$:

   - definitely converges.
   - definitely diverges.
   - may or may not converge.

2. What would be a good series to compare $\sum_{k=1}^{\infty} \frac{\sqrt{3k^2+k+1}}{2k^2+3k+4}$ to if using the Limit Comparison Test? Explain.

3. Determine whether the following series converge or diverge. Be sure to explain your reasoning!

(a) $\displaystyle\sum_{n=4}^{\infty} \frac{5}{n\sqrt[3]{n}}$

(b) $\displaystyle\sum_{n=4}^{\infty} \frac{10n+1}{(2n^2+1)\sqrt[3]{n-2}}$

4. Consider the series: $\displaystyle\sum_{k=1}^{\infty} \frac{\tan^{-1}(k^2)}{k^3}$.

(a) Explain why for $k \geq 1$, $0 < \tan^{-1}(k^2) < \frac{\pi}{2}$ and show $0 < \dfrac{\tan^{-1}(k^2)}{k^3} < \dfrac{\pi}{2k^3}$.

(b) Does $\displaystyle\sum_{k=1}^{\infty} \frac{\pi}{2k^3}$ converge or diverge? Explain.

(c) What, if anything, can you conclude about $\displaystyle\sum_{k=1}^{\infty} \frac{\tan^{-1}(k^2)}{k^3}$ in light of parts (a) and (b)?

5. (a) Find the equation of the tangent line to $y = \sin(\theta)$ at $(0, 0)$.

(b) Use your answer to part (a) to explain why if $\theta \approx 0$, then $\sin(\theta) \approx \theta$.

(c) Use part (b) to explain why $\sin\left(\frac{1}{x\sqrt[3]{x}}\right) \approx \frac{1}{x\sqrt[3]{x}}$ as $x \to \infty$.

(d) Use part (c) to find a good candidate to which to compare the series: $\displaystyle\sum_{k=1}^{\infty} \sin\left(\frac{1}{k\sqrt[3]{k}}\right)$.

(e) Determine whether or not $\displaystyle\sum_{k=1}^{\infty} \sin\left(\frac{1}{k\sqrt[3]{k}}\right)$ converges using the Limit Comparison Test.

NOTE: Make sure you pass to a continuous variable, such as x, before applying L'Hopital's Rule ...

SECTION 10.6 PRACTICE PROBLEMS

1. List the three conditions that must be met in order to apply the Alternating Series Test to an infinite series:

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   -
   -

2. In order to show a series $\sum_{k=1}^{\infty} a_k$ is **conditionally convergent**, what two things need to be shown?

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3. (a) Prove the series $\sum_{k=1}^{\infty} \frac{4\,(-1)^{k+1}}{2k-1}$ converges using the Alternating Series Test (AST).

NOTE: There is no exponent on ‘4’.

(b) Recall the AST Remainder Theorem: If $\sum_{k=1}^{\infty} a_k$ converges by the AST and we define:

$$S_n = \sum_{k=1}^{n} a_k \qquad \text{and} \qquad R_n = \sum_{k=n+1}^{\infty} a_k,$$

Then: $|R_n| \leq |a_{n+1}|$.

Use the AST Remainder Theorem to find n so that $|R_n| \leq 0.0005$.

(c) Find $\displaystyle\sum_{k=1}^{\infty} \frac{4\,(-1)^{k+1}}{2k-1}$ to within an error of 0.0005 using the corresponding sum, S_n from part (b).

NOTE: Feel free to use Desmos or a graphing calculator to do the heavy lifting here!

(d) Prove $\displaystyle\sum_{k=1}^{\infty} \frac{4\,(-1)^{k+1}}{2k-1}$ is **conditionally** convergent.

SECTION 10.7 PRACTICE PROBLEMS

1. The ratio and root tests are checking to see if a given series is 'close to' what type of infinite series?

2. For each of the given scenarios below, circle the most correct answer.

   (a) If $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \frac{5}{4}$, then the series $\sum_{k=1}^{\infty} a_k$:

   - definitely converges.
   - definitely diverges.
   - may or may not converge.

   (b) If $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1$, then the series $\sum_{k=1}^{\infty} a_k$:

   - definitely converges.
   - definitely diverges.
   - may or may not converge.

   (c) If $\lim_{k\to\infty}\sqrt[k]{|a_k|} = \frac{4}{5}$, then the series $\sum_{k=1}^{\infty} a_k$:

   - definitely converges.
   - definitely diverges.
   - may or may not converge.

   (d) If $\lim_{k\to\infty}\sqrt[k]{|a_k|} = \frac{4}{3}$, then the series $\sum_{k=1}^{\infty} a_k$:

   - definitely converges.
   - definitely diverges.
   - may or may not converge.

3. (a) Explain why the series $\displaystyle\sum_{k=1}^{\infty} \frac{117}{k^{1.0001}}$ is convergent.

(b) Prove the series $\displaystyle\sum_{k=1}^{\infty} \frac{117\cos(12\ln(k))}{k^{1.0001}}$ is absolutely convergent.

HINT: $|\cos(\theta)| \leq 1 \ldots$

4. Determine whether the following series converge or diverge. Be sure to explain your reasoning!

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{42^k}{k!}$

(b) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{3k^2+2k-1}{2k^2+6k+3}\right)^{3k}$

5. Let $a_k = \dfrac{1}{1 \cdot 3 \cdot \ldots \cdot (2k-1)}$ for $k \geq 1$. To be clear, this means:

$$a_1 = \frac{1}{1} = 1, \qquad a_2 = \frac{1}{1 \cdot 3} = \frac{1}{3}, \qquad a_3 = \frac{1}{1 \cdot 3 \cdot 5} = \frac{1}{15}, \qquad a_4 = \frac{1}{1 \cdot 3 \cdot 5 \cdot 7} = \frac{1}{105}$$

(a) Find an expression for a_{k+1} then find and simplify an expression for the ratio $\dfrac{a_{k+1}}{a_k}$.

(b) Does the series $\displaystyle\sum_{k=1}^{\infty} a_k$ converge or diverge? Explain your reasoning.

(c) Determine if $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{1 \cdot 3 \cdot \ldots \cdot (2k-1)}$ converges or diverges. Explain your reasoning.