MATH 2500: TAKE HOME 02 (20 POINTS)

NAME: ____

DIRECTIONS: Make sure your work is neat and complete and uses the techniques demonstrated in class.

SECTION 3.1 PRACTICE PROBLEMS

1. Fill-in the blanks below:

   (a) $f(a)$ gives you:

   - the ____________________ of the function with **input** $x = a$.
   - the ____________________-value on the graph of $y = f(x)$ at $x = a$.

   (b) $f'(a)$ gives you:

   - the ____ of outputs to inputs at $x = a$.
   - the ____ on the graph of $y = f(x)$ at $x = a$.

2. For $f(x) = \sqrt{4x+1}$:

(a) Find the slope of the tangent line at $(2, f(2))$ using: $m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

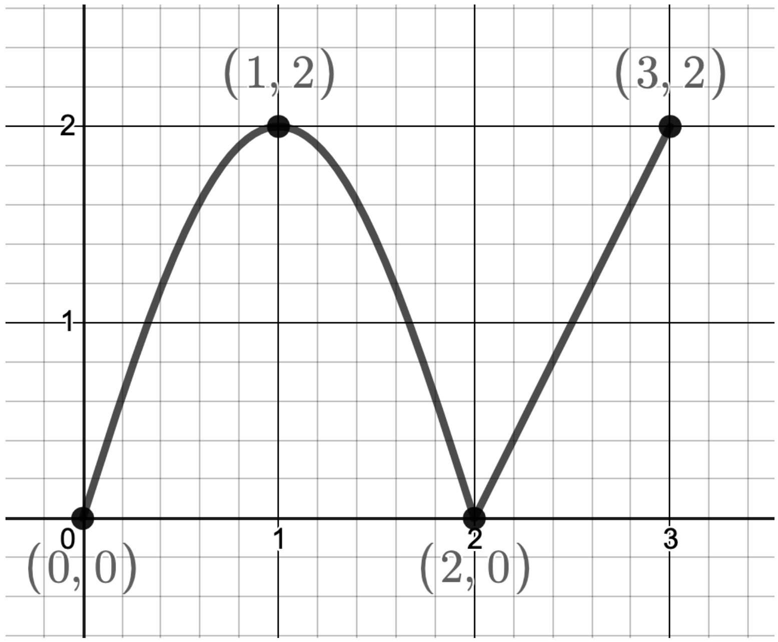
(b) Write the equation of the tangent line at $(2, f(2))$. Check your answer graphically.

3. For $f(x) = \ln(x)$:

(a) **Approximate** the slope of the tangent line at $(2, f(2))$ by **approximating**: $m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

(b) Write the equation of the tangent line at $(2, f(2)))$. Check your answer graphically.

4. Use the graph of $y = f(x)$ below to answer the following questions and explain your reasoning.



(a) Find $f'(2.5)$ and explain your reasoning.

(b) Where is $f'(x) = 0$?

(c) Explain why f is not differentiable at $x = 2$.

(d) Is $f(1.5)$ positive or negative? What about $f'(1.5)$? Explain your reasoning.

(e) Which is larger: $f'(0.5)$ or $f'(0.9)$? Explain your reasoning.

SECTION 3.2 PRACTICE PROBLEMS

1. Let $f(x) = \sqrt{4x+1}$.

   (a) Use the limit definition of derivative: $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to find a formula for $f'(x)$.

   (b) Use part (a) to help you find the equation of the tangent line to $y = \sqrt{4x+1}$ at $(2, 3)$.

   Check your answer using a graphing utility (and your answer to 2.)

2. Let $f(x) = \dfrac{6}{2x+3}$.

(a) Use the limit definition of derivative: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to find a formula for $f'(x)$.

(b) Use part (a) to help you find the equation of the tangent line to $y = \dfrac{6}{2x+3}$ at $(0, 2)$.

Check your answer using a graphing utility.

3. Let $f(x) = \cos(x)$.

(a) Use the limit definition of derivative: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to find a formula for $f'(x)$.

RECALL: $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$, $\lim_{h \to 0} \dfrac{\sin(h)}{h} = 1$, and $\lim_{h \to 0} \dfrac{\cos(h) - 1}{h} = 0$.

(b) Use part (a) to help you find the equation of the tangent line to $y = \cos(x)$ at $\left(\frac{\pi}{2}, 0\right)$

Check your answer using a graphing utility.

4. Sketch the graph of a continuous function where $f(3) = 2$ and $\lim_{x \to 3} \dfrac{f(x) - f(3)}{x - 3} = -\infty$.

Explain your reasoning.

SECTION 3.3 and 3.4 PRACTICE PROBLEMS

1. Find the indicated derivative.

   (a) $D_x\left[2x^3 - 6x + 1\right]$

   (b) For $f(x) = 3x^2 - 4\sqrt{x} + \dfrac{2}{x}$, find $f'(x)$.

   (c) For $y = \dfrac{3x^2 - 2x + 1}{4x^3}$, find y'.

   (d) If $y = \dfrac{3 - 4x}{2x + 1}$, find and simplify $\dfrac{dy}{dx}$.

2. Find the equation of the tangent line to the graph of the given function at the indicated value.

Check your answer using a graphing utility.

(a) $f(x) = 2x^2 - 3x + 1$ at $x = -1$.

(b) $g(t) = \dfrac{4}{\sqrt[4]{t}}$ at $t = 16$.

(c) $F(s) = \dfrac{s}{s^2 + 1}$ at $s = \sqrt{3}$.

3. Find and simplify the indicated derivative.

(a) For $f(x) = \dfrac{3x - 1}{\sqrt{x}}$, find $f''(x)$.

(b) Find $D_t^3 \left[4t^3 - 3t^2 + 2t - 1\right]$.

SECTION 3.5 PRACTICE PROBLEMS

1. Find the indicated derivative.

    (a) For $f(x) = 2x\sin(x)$, find $f'(x)$.

    (b) If $y = \tan(t) - 3\sec(t)$, find y'.

    (c) Find $D_t\left[\sin(t) - t\cos(t) + 2\right]$.

    (d) If $S(\theta) = \sec(\theta)$, find $S''(\theta)$.

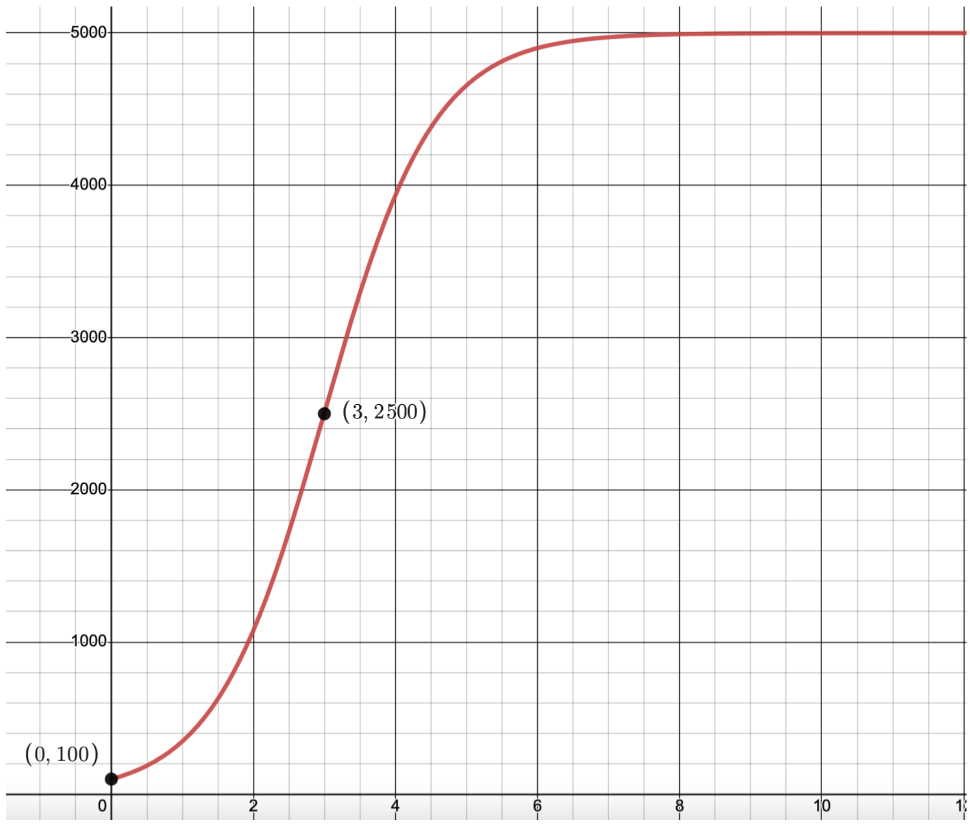
2. Let $f(x) = x\cos(x)$.

(a) Write the equation of the line tangent to $y = f(x)$ at $x = \dfrac{\pi}{2}$.

(b) Check your answer graphically. Sketch the graph.

SECTION 3.6 PRACTICE PROBLEMS

1. The graph of the function P below plots the number of bacteria t hours after it is introduced into a medium.



$y = P(t)$

(a) How many bacteria were initially introduced to the medium?

(b) When does the population of bacteria seem to be growing the fastest? Explain using the graph.

(c) What appears to be $\lim_{t\to\infty} P(t)$. What does this mean in terms of time and the population of bacteria?

2. Suppose a ball is thrown into the air and its height off of the ground $s(t)$, in feet, t seconds after it is tossed into the air is given by: $s(t) = -16t^2 + 16t + 192$.

(a) Find and interpret $v(1)$ and $a(1)$. Is the ball speeding up or slowing down when $t = 1$? Explain.

(b) Find when $v(t) = 0$ and interpret your answer. How high off of the ground is the ball at this point?

(c) How fast is the ball traveling when it strikes the ground?

3. Suppose the cost, in dollars, to make x items is $C(x) = x^2 + 400x + 250$.

(a) Find and interpret $C(25)$, $\overline{C}(25)$, and $MC(25)$.

$C(25) =$ ______________ and means:

$\overline{C}(25) =$ ______________ and means:

$MC(25) =$ ______________ and means:

(b) Compare $\overline{C}(25)$ and $MC(25)$. Would producing an additional item lower the average cost? Explain.

(c) Calculate $\overline{C}(26)$ and compare your answer to $\overline{C}(25)$ to check your answer to part (b).

4. Suppose $P(t)$ gives the population of a certain species of turtles in Lake County t years after 2015.

(a) Interpret the following equations in terms of time and turtles:

i. $P(3) = 25000$.

ii. $P'(3) = -250$

(b) Write a mathematical equation equivalent to:

i. In 2023, there are 30000 turtles in Lake County.

ii. In 2023, the turtle population is rising at a rate of 50 turtles per year.

iii. Using parts (i) and (ii), estimate the turtle population in Lake Country in 2025.

SECTION 3.7 PRACTICE PROBLEMS

1. Fill in the blanks below.

   (a) For $h(x) = \sin(2x)$, we may write $h(x) = f(g(x))$ where $f(x) = \sin(x)$ and $g(x) =$ ____________________.

   (b) For $h(x) = \dfrac{3}{2x+5}$, we may write $h(x) = f(g(x))$ with $g(x) = 2x+5$ and $f(x) =$ ____________________.

   (c) The differentiation rule which governs function composition is the ____________________ Rule.

   It states: $D_x[f(g(x))] =$ ____.

2. Find and simplify the indicated derivative.

(a) $D_x\left[(x^2+1)^{17}\right]$

(b) Find $f'(x)$ for $f(x) = \dfrac{5}{\sqrt{2-x}}$. **HINT:** Rewrite to avoid the quotient rule ...

3. Find and simplify the indicated derivative.

(a) For $y = \sec(5t)$, find y''.

(b) If $y = \tan(5t)$, find $\frac{d^2y}{dt^2}$.

4. Suppose $f(2) = -5$ and $f'(2) = 3$. Find the value of $F'(2)$ for the following functions $F(x)$:

(a) $F(x) = x^2 f(x)$

(b) $F(x) = \dfrac{f(x)}{3x - 1}$

(c) $F(x) = \sqrt[3]{f(x)}$

(d) $F(x) = f(8 - 3x)$

(e) $F(x) = f\left(\dfrac{4}{x}\right)$.

SECTION 3.8 PRACTICE PROBLEMS

1. Let C be the curve in the plane described by the equation: $x^3 + y^3 = (x^2 + y^2)^2$

   (a) Algebraically show the points $(0,0)$, $(0,1)$, $(1,0)$, and $\left(\frac{1}{2}, \frac{1}{2}\right)$ are on C.

   - $(0,0)$:
   - $(0,1)$:
   - $(1,0)$:
   - $\left(\frac{1}{2}, \frac{1}{2}\right)$:

   (b) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

(c) Find $\dfrac{dy}{dx}$ at each of the points $(0,1)$, and $\left(\dfrac{1}{2},\dfrac{1}{2}\right)$:

- $\left.\dfrac{dy}{dx}\right|_{(x,y)=(0,1)} =$

- $\left.\dfrac{dy}{dx}\right|_{(x,y)=\left(\frac{1}{2},\frac{1}{2}\right)} =$

(d) Find the equation of the tangent line at each of the points $(0,1)$, and $\left(\dfrac{1}{2},\dfrac{1}{2}\right)$:

- Tangent line at $(x,y)=(0,1)$:

- Tangent line at $(x,y)=\left(\dfrac{1}{2},\dfrac{1}{2}\right)$:

(e) Check your answer to part 1d using desmos.

(f)
- What happens when you try to find $\left.\dfrac{dy}{dx}\right|_{(x,y)=(0,0)}$?

- Zoom in near $(0,0)$. What appears to be $\left.\dfrac{dy}{dx}\right|_{(x,y)=(0,0)}$?

- What appears to be the equation of the tangent line at $(0,0)$? Check your answer using desmos.

(g)
- What happens when you try to find $\left.\dfrac{dy}{dx}\right|_{(x,y)=(1,0)}$?

- Zoom in near $(1,0)$. What feature does there appear to be?

- What appears to be the equation of the tangent line at $(1,0)$? Check your answer using desmos.

SECTION 3.9 PRACTICE PROBLEMS

1. Team Rocket's hot air balloon is ascending vertically from a point on level ground at a constant rate of 6 feet per second. Let θ be the angle of inclination to the base of the balloon basket from a point on the ground 40 feet away from the launch point. How fast is θ changing when the balloon is 30 feet off the ground? Be sure to adequately explain the units on your answer.

2. It takes 2 minutes for the 160 foot Ashtabula Bascule Lift Bridge to rotate 45° from its horizontal position to its raised position, as seen below.

You can see a video of the bridge being raised here: http://www.youtube.com/watch?v=PRwZzzPyK2g

Assuming the bridge casts a shadow directly below itself the entire time it is being raised,[1] find the rate at which the tip of the bridge's shadow is receding at the instant the bridge has rotated 30°.

HINT: You may assume the angle of elevation of the bridge changes at a constant rate.

[1]That is, the sun is directly overhead of the bridge and is shining for an entire two minutes... which never actually happens.

EXPLORATION Find the indicated derivative. What pattern emerges?

- $D_x[x]$
- $D_x^2[x^2]$
- $D_x^3[x^3]$
- $D_x^4[x^4]$
- $D_x^5[x^5]$
- $D_x^6[x^6]$
- Based on your observations, what is $D_x^n[x^n]$ if $n = 1, 2, 3, ...$?

EXPLORATION: We can use implicit differentiation to help us find the derivatives of inverse functions!

Let $y = \tan^{-1}(x)$ (also called 'arctan(x)'.).

- From $y = \tan^{-1}(x)$, we get $\tan(y) = x$.

  Implicitly differentiate $\tan(y) = x$ with respect to x to show: $\dfrac{dy}{dx} = \dfrac{1}{\sec^2(y)}$.

- Use an identity to rewrite $\sec^2(y)$ in terms of $\tan(y)$, and hence, x.

- Put part (a) and (b) together to show $D_x\left[\tan^{-1}(x)\right] = \dfrac{1}{x^2+1}$.

EXTRA (EXTRA) PRACTICE WITH DERIVATIVES

Find and simplify the indicated derivative. Use Desmos to check your answers graphically!

1. $D_x\left[3x^2 - 2x + 1\right]$

2. $\dfrac{d}{dx}\left[\dfrac{2x-1}{3\sqrt{x}}\right]$

3. $f'(x)$ if $f(x) = \dfrac{3x+2}{x^2-7}$

4. $\dfrac{dF}{ds}$ if $F(s) = \dfrac{3}{s^2+9}$

5. y'' if $y = \dfrac{5x-3}{2x+1}$

6. $\dfrac{dy}{dx}$ if $y = x\sqrt[3]{1-9x}$

7. y' if $y = \sin(x) - \cos(x)$

8. $g'(x)$ if $g(x) = \sec\left(x^2\right)$

9. $\dfrac{d}{dt}\left[t^2\cos(3t)\right]$

10. $D_\theta^2\left[\cot(2\theta)\right]$

11. $F''(\theta)$ if $F(\theta) = \dfrac{\cos(\theta)}{1-\sin(\theta)}$

12. $f'(x)$ if $f(x) = \dfrac{x\sqrt{4x-3}}{2x+1}$

EXTRA (EXTRA) PRACTICE WITH DERIVATIVES - ANSWERS

1. $D_x\left[3x^2 - 2x + 1\right] = 6x - 2$

2. $\dfrac{d}{dx}\left[\dfrac{2x-1}{3\sqrt{x}}\right] = \ldots = \dfrac{d}{dx}\left[\dfrac{2}{3}\,x^{1/2} - \dfrac{1}{3}\,x^{-1/2}\right] = \dfrac{1}{3}\,x^{-1/2} + \dfrac{1}{6}\,x^{-3/2} = \ldots = \dfrac{2x+1}{6\,x^{3/2}}$

3. $f'(x) = \dfrac{(x^2-7)(3) - (3x+2)(2x)}{(x^2-7)^2} = \ldots = -\dfrac{3x^2+4x+21}{(x^2-7)^2}$

4. $\dfrac{dF}{ds} = D_s\left[3(s^2+9)^{-1}\right] = \ldots = -\dfrac{6s}{(s^2+9)^2}$

5. $y' = \dfrac{11}{(2x+1)^2}$, so $y'' = D_x\left[\dfrac{11}{(2x+1)^2}\right] = D_x\left[11(2x+1)^{-2}\right] = \ldots = -\dfrac{44}{(2x+1)^{-3}}$.

6. $\dfrac{dy}{dx} = D_x\left[x\,(1-9x)^{1/3}\right] = \ldots = (1-9x)^{1/3} - \dfrac{3x}{(1-9x)^{2/3}} = \ldots = \dfrac{1-12x}{(1-9x)^{2/3}}$

7. $y' = \cos(x) + \sin(x)$

8. $g'(x) = 2x\sec\left(x^2\right)\tan\left(x^2\right)$

9. $\dfrac{d}{dt}\left[t^2\,\cos(3t)\right] = 2t\cos(3t) - 3t^2\sin(3t)$

10. $D_\theta\left[\cot(2\theta)\right] = -2\csc^2(2\theta) = -2\left(\csc(2\theta)\right)^2$, so $D_\theta^2\left[\cot(2\theta)\right] = D_\theta\left[-2\left(\csc(2\theta)\right)^2\right] = \ldots = 8\csc^2(2\theta)\cot(2\theta)$

11. $F'(\theta) = \ldots = \dfrac{-\sin(\theta) + \sin^2(\theta) + \cos^2(\theta)}{(1-\sin(\theta))^2} = \ldots = \dfrac{1}{1-\sin(\theta)} = (1-\sin(\theta))^{-1}$ so $F''(\theta) = \dfrac{\cos(\theta)}{(1-\sin(\theta))^2}$

12. $f'(x) = \ldots = \dfrac{(2x+1)[(4x-3)^{1/2} + 2x(4x-3)^{-1/2}] - 2x(4x-3)^{1/2}}{(2x+1)^2} = \ldots = \dfrac{4x^2+6x-3}{(2x+1)^2(4x-3)^{1/2}}$