MATH 2500: TAKE HOME 01 (20 POINTS)

NAME: ____________________

DIRECTIONS: Make sure your work is neat and complete and uses the techniques demonstrated in class.

SECTION 2.2 PRACTICE PROBLEMS

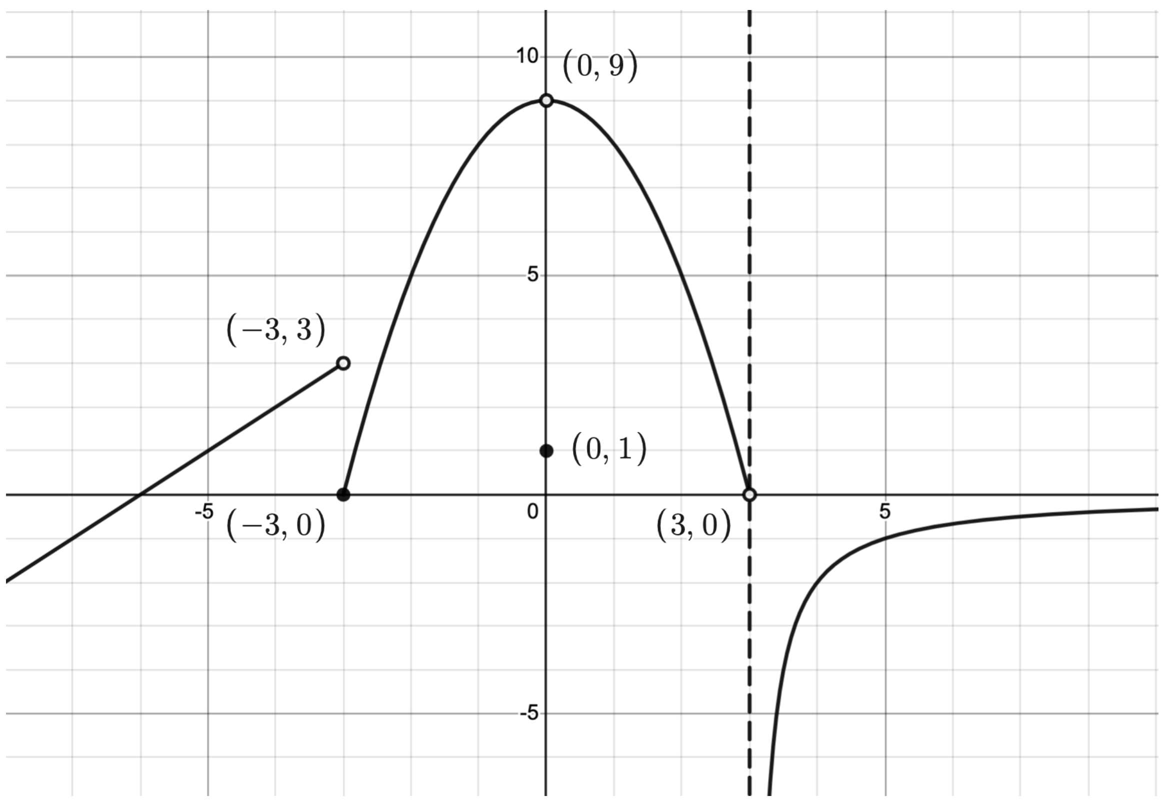
1. Circle the correct choice: given a function, f:

   $f(a)$ is what we **get** / **expect to get** at $x = a$, whereas $\lim_{x\to a} f(x)$ is what we **get** / **expect to get** at $x = a$.

2. Consider the complete graph of the function f below. Use the graph to find the indicated values.

   If a limit fails to exist, write 'd.n.e.' or use the symbols '∞' or '$-\infty$' appropriately.

   NOTE: The graph has a vertical asymptote $x = 3$.



- $\lim_{x\to -3^-} f(x)$
- $\lim_{x\to -3^+} f(x)$
- $\lim_{x\to -3} f(x)$
- $f(-3)$
- $\lim_{x\to 0} f(x)$
- $f(0)$
- $\lim_{x\to 3^-} f(x)$
- $\lim_{x\to 3^+} f(x)$

Change just one point on the graph so that $\lim_{x\to 0} f(x) = f(0)$.

3. (a) Consider $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$. Let $f(x) = \sin\left(\frac{\pi}{x}\right)$ and fill-in the tables below.

Table A	
x	$f(x)$
-0.001	
-0.0001	
-0.00001	
-0.000001	
0.000001	
0.00001	
0.0001	
0.001	

Table B	
x	$f(x)$
$-\frac{2}{1001}$	
$-\frac{2}{1005}$	
$-\frac{2}{1009}$	
$-\frac{2}{1013}$	
$\frac{2}{1015}$	
$\frac{2}{1011}$	
$\frac{2}{1007}$	
$\frac{2}{1003}$	

(b) Based on Table A, what *appears* to be $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$? Explain your reasoning.

(c) Based on Table B, what *appears* to be $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$? Explain your reasoning.

(d) Looking at the data from Table A and Table B together what do you suspect about $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$?

Confirm your answer graphically.

(e) Find a sequence of real numbers, x so that $x \to 0$ and $f(x) = 1$.

4. Consider the table of values below:

x	$f(x)$
-0.001	1.9
-0.0001	1.99
-0.00001	1.999
-0.000001	1.9999
0.000001	-10000
0.00001	-1000
0.0001	-100
0.001	-10

It turns out that $\lim_{x \to 0} f(x) = 117$. How is this possible assuming the data in the table is correct?

5. (a) Explain why $\lim_{x \to 5} \sqrt{5-x} \neq 0$ but $\lim_{x \to 5} \sqrt[3]{5-x} = 0$.

(b) Explain why $\lim_{x \to 5^-} \sqrt{5-x} = 0$.

6. **EXPLORATION:** Let $f(x) = \lfloor 2\sin(x) \rfloor$.

(a) Let's analyze f near $x = \dfrac{\pi}{6}$.

i. Find the following values analytically:

- $f\left(\frac{\pi}{6}\right)$.
- $\lim\limits_{x \to \frac{\pi}{6}^-} f(x)$.
- $\lim\limits_{x \to \frac{\pi}{6}^+} f(x)$.

ii. Based on your answers, sketch the graph of $y = f(x)$ near $x = \dfrac{\pi}{6}$.

iii. Compare your answers with what desmos gives. What do you notice?

(b) Let's analyze f near $x = \dfrac{\pi}{2}$.

i. Find the following values analytically:

- $f\left(\frac{\pi}{2}\right)$.
- $\lim\limits_{x \to \frac{\pi}{2}^-} f(x)$.
- $\lim\limits_{x \to \frac{\pi}{2}^+} f(x)$.

ii. Based on your answers, sketch the graph of $y = f(x)$ near $x = \dfrac{\pi}{2}$.

iii. Compare your answers with what desmos gives. What do you notice?

SECTION 2.3 PRACTICE PROBLEMS

1. Determine the following limits analytically using the methods demonstrated in class.

   If a limit does not exist, write 'd.n.e.'

   (a) $\lim\limits_{x \to 2} \dfrac{2x^2 + x - 3}{x^2 - 1}$

   (b) $\lim\limits_{x \to 1} \dfrac{2x^2 + x - 3}{x^2 - 1}$

   (c) $\lim\limits_{x \to -1} \dfrac{2x^2 + x - 3}{x^2 - 1}$

(d) $\lim_{x \to -1} \frac{\sqrt{x+5}-2}{x+1}$

(e) $\lim_{h \to 0} \frac{\sqrt{2x+2h-1}-\sqrt{2x-1}}{h}$

(f) $\lim_{x \to 2} \frac{\frac{2x}{x+2}-1}{x-2}$

(g) $\lim_{h \to 0} \frac{\frac{1}{2(x+h)-1}-\frac{1}{2x-1}}{h}$

(h) $\lim_{x \to 3^-} \frac{|3x - x^2|}{x - 3}$

(i) $\lim_{x \to 2^+} \frac{|6 - 3x|}{x^2 - 4x + 4}$

(j) $\lim_{x \to \pi} \tan\left(\frac{3x}{4}\right)$

(k) $\lim_{x \to 0} (2x \csc(3x))$

2. (a) Graph $f(x) = |x| \cos\left(\frac{3}{x}\right)$ using desmos. What appears to be $\lim_{x \to 0} |x| \cos\left(\frac{3}{x}\right)$?

Include a sketch to explain your reasoning.

(b) Explain what is wrong with the following argument:

$$
\begin{aligned}
\lim_{x \to 0} |x| \cos\left(\frac{3}{x}\right) &= \lim_{x \to 0} |x| \cdot \lim_{x \to 0} \cos\left(\frac{3}{x}\right) && \text{Reason: Product Rule} \\
&= 0 \cdot \lim_{x \to 0} \cos\left(\frac{3}{x}\right) && \text{Reason: Since } \lim_{x \to 0} |x| = 0. \\
&= 0 && \text{Since } 0 \cdot \text{anything} = 0.
\end{aligned}
$$

HINT: What must be true to use limit properties?

(c) Use the fact that $-1 \leq \cos(\theta) \leq 1$ for all values of θ to show: $-|x| \leq |x| \cos\left(\frac{3}{x}\right) \leq |x|$ for all $x \neq 0$.

Verify this inequality graphically using desmos.

(d) Use the Squeeze Theorem to help you find $\lim_{x \to 0} |x| \cos\left(\frac{3}{x}\right)$.

SECTION 2.4 / 2.5 PRACTICE PROBLEMS

1. Let $f(x) = \dfrac{1-2x}{x-5}$. Circle the correct response or fill in the blank, as required.

   (a) For $\lim\limits_{x \to 5^-} f(x)$, we are considering x-values that are a little bit **greater** / **less** than $x = 5$.

   (b) The numerator of $f(x)$, $1 - 2x$, approaches ______ as $x \to 5^-$.

   (c) The denominator of $f(x)$, $x - 5$, becomes a very 'small' **positive** / **negative** number as $x \to 5^-$.

   (d) Hence, $f(x) = \dfrac{1-2x}{x-5}$ becomes very 'large' **positive** / **negative** number so we write $\lim\limits_{x \to 5^-} f(x) =$______.

   (e) What graphical feature do we expect to see at $x = 5$? Use desmos to check your answer.

2. Consider the limit: $\lim_{x\to 2^+} \dfrac{x^2 - x - 2}{x^2 - 4x + 4}$

(a) Show that direct substitution results in the indeterminate form, '$\dfrac{0}{0}$'.

(b) Find $\lim_{x\to 2^+} \dfrac{x^2 - x - 2}{x^2 - 4x + 4}$ analytically. Use the symbols '∞' and '$-\infty$' as appropriate.

(c) Do '$\dfrac{0}{0}$' indeterminate forms always resolve themselves to produce a real number? Explain.

3. **EXPLORATION:** Let $f(x) = \dfrac{x}{\lfloor x \rfloor}$. Fill in the blanks below to help you analyze $\lim_{x \to 0} f(x)$.

(a) If $-1 < x < 0$, then $\lfloor x \rfloor =$ ______________. So we can rewrite $f(x) = \dfrac{x}{\lfloor x \rfloor} =$ ______________.

(b) Using part (a), we can find $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}$ ______________ $=$ ______________.

(c) If $0 < x < 1$, then $\lfloor x \rfloor =$ ______________, hence $f(x) = \dfrac{x}{\lfloor x \rfloor}$ is ______________ as $x \to 0^+$.

(d) Putting parts (b) and (c) together, we have that $\lim_{x \to 0} f(x)$ ____________________

(e) Graph $f(x) = \dfrac{x}{\lfloor x \rfloor}$ on desmos near $x = 0$ to confirm your answers.

4. Consider the limit: $\lim_{x \to \infty} \frac{1-2x}{x-5}$.

(a) Use the Leading Term Test to predict the value of $\lim_{x \to \infty} \frac{1-2x}{x-5}$.

(b) Interpret what your answer to part (a) means graphically and check your answer using desmos.

(c) Analytically determine $\lim_{x \to \infty} \frac{1-2x}{x-5}$ using the method demonstrated in class.

5. Determine the following limits analytically using the methods demonstrated in class.

   Use the symbols '∞' and '$-\infty$' as appropriate.

   (a) $\displaystyle\lim_{x\to\infty} \frac{4-x}{x^2+2x+3}$.

   (b) What does your answer to part (a) tell you about the graph of the function in part (a)?

   (c) $\displaystyle\lim_{x\to\infty} \frac{x^2+2x+3}{4-x}$.

   (d) Explain how you know the graph of the function in part (c) has a slant asymptote and find it.

6. Let $f(x) = \sqrt{25x^2 - 10x + 4}$.

(a) Use the Leading Term Test to show that as $x \to \infty$, $f(x) \approx 5x$.

(b) Use desmos to graph $y = f(x)$ and $y = 5x$.

Is $y = 5x$ a slant asymptote to the graph of $y = f(x)$ as $x \to \infty$? Explain.

(c) Find $\lim_{x \to \infty} \left(\sqrt{25x^2 - 10x + 4} - 5x \right)$ to help you find the slant asymptote of $y = f(x)$ as $x \to \infty$.

Check your answer using desmos.

SECTION 2.6 PRACTICE PROBLEMS

1. Fill in the blanks:

   (a) A function f is continuous at $x = a$ if $\lim_{x \to a} f(x) =$ ______.

   (b) This means, you **get** what you __________ at $x = a$.

   (c) For a continuous function, inputs that are 'close' will result in __________ that are 'close.'

   (d) Sketch a picture of each scenario below. Which type of discontinuity is present?

   - $f(a)$ does not exist but $\lim_{x \to a} f(x)$ exists.

   - $f(a)$ exists but $\lim_{x \to a} f(x)$ does not exist.

   - $f(a)$ and $\lim_{x \to a} f(x)$ exist but $f(a) \neq \lim_{x \to a} f(x)$.

2. Let $f(x) = \dfrac{2x^2 - 3x - 2}{x^2 + 3x - 10}$.

(a) Determine the x-values where f is not continuous.

(b) Use limits to analyze f near each discontinuity. Use the symbols '∞' and '$-\infty$' as appropriate.

(c) If there any removable discontinuities, define f so as to remove them.

3. Let $C(x) = x\cot(7x)$.

(a) Explain why C is not continuous at $x = 0$.

(b) Find $\lim_{x\to 0} C(x)$ analytically by following the steps below:

i. Use the fact that $\lim_{t\to 0} \frac{\sin(t)}{t} = 1$ to find $\lim_{x\to 0} \frac{\sin(7x)}{x}$ using the substitution $t = 7x$.

HINT: If $t = 7x$, then $x = \frac{t}{7}$ and $x \to 0$ if and only if $t \to 0$.

ii. Use your answer for $\lim_{x\to 0} \frac{\sin(7x)}{x}$ to help you find $\lim_{x\to 0} \frac{x}{\sin(7x)}$.

iii. Use a trigonometric identity, your answer to (ii), and the product rule to help you find $\lim_{x\to 0} C(x)$.

(c) Explain why the discontinuity at $x = 0$ is removable.

Define $C(0)$ so as to remove the discontinuity:

$$C(x) = \begin{cases} \underline{\hspace{3cm}}, & \text{if } x \neq 0 \\ \\ \underline{\hspace{3cm}}, & \text{if } x = 0 \end{cases}$$

(d) Sketch the graph of $y = C(x)$ before and after 'the patch.'

$y = C(x)$ 'near' $x = 0$ 'pre-patch.'

$y = C(x)$ 'near' $x = 0$ 'post-patch.'

4. Sketch the graph of a function f which satisfies all of the following criteria:

- $\lim_{x \to -\infty} f(x) = 2$
- $\lim_{x \to 0^-} f(x) = \infty$
- $\lim_{x \to 0^+} f(x) = -\infty$
- $\lim_{x \to 2^-} f(x) = 3$
- $\lim_{x \to 2^+} f(x) = 0$
- $\lim_{x \to \infty} f(x) = -\infty$
- The intervals of continuity of f are $(-\infty, 0)$, $(0, 2]$, $(2, \infty)$.
- The graph of f has a slant asymptote $y = 5 - x$.

5. Let $f(x) = x^5 - x + 1$.

(a) Find $f(-2)$ and $f(-1)$.

$f(-2) =$ ______________________ $f(-1) =$ ______________________

(b) Explain how we know f is continuous on $[-2, 1]$.

(c) Which theorem guarantees there is at least one x-intercept between $x = -2$ and $x = -1$?

Use desmos to approximate the x-coordinate of the x-intercept, rounded to four decimal places.

(d) Explain how parts (a) through (c) guarantees a solution to $x^5 + 1 = x$ between $x = -2$ and $x = -1$.

6. **EXPLORATION:** A real number x is said to be a **fixed point** of a function f if $f(x) = x$.

(a) Show $x = 3$ is a fixed point of $f(x) = \dfrac{9}{x}$ and find all the other fixed points of f.

Check your answers using desmos.

(b) Fill in the reasoning for the statements below which proves the following **THEOREM**:

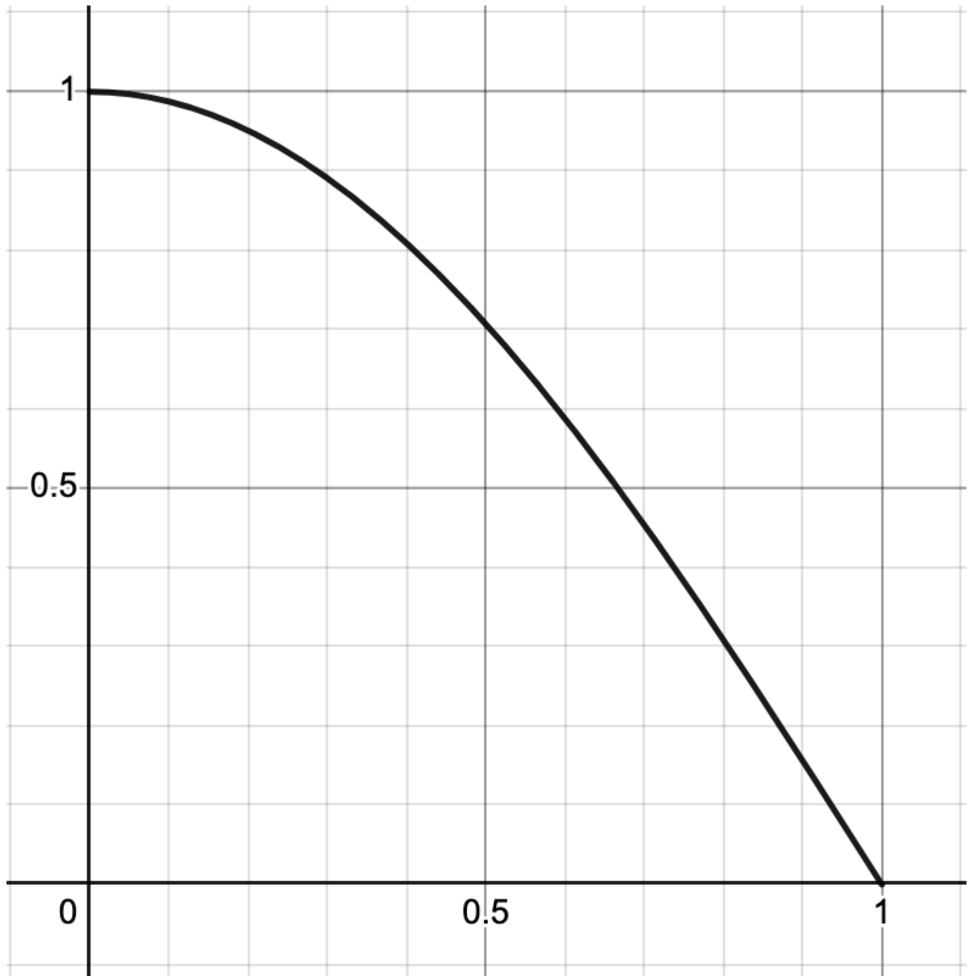
IF f is **continuous** on $[a, b]$, $f(a) > a$, AND $f(b) < b$, THEN f has at least one fixed point in (a, b).

PROOF: Let $g(x) = f(x) - x$.

STATEMENT:	**REASON:**
The function g is continuous on $[a, b]$.	____________________
$g(a) > 0$ and $g(b) < 0$.	____________________
There exists x between a and b with $g(x) = 0$.	____________________
Hence $f(x) = x$ so x is a fixed point.	____________________

(c) Below is the graph of a continuous function f which is continuous on $[0, 1]$ with $f(0) > 0$ and $f(1) < 1$.

Use the graph to estimate the fixed point.

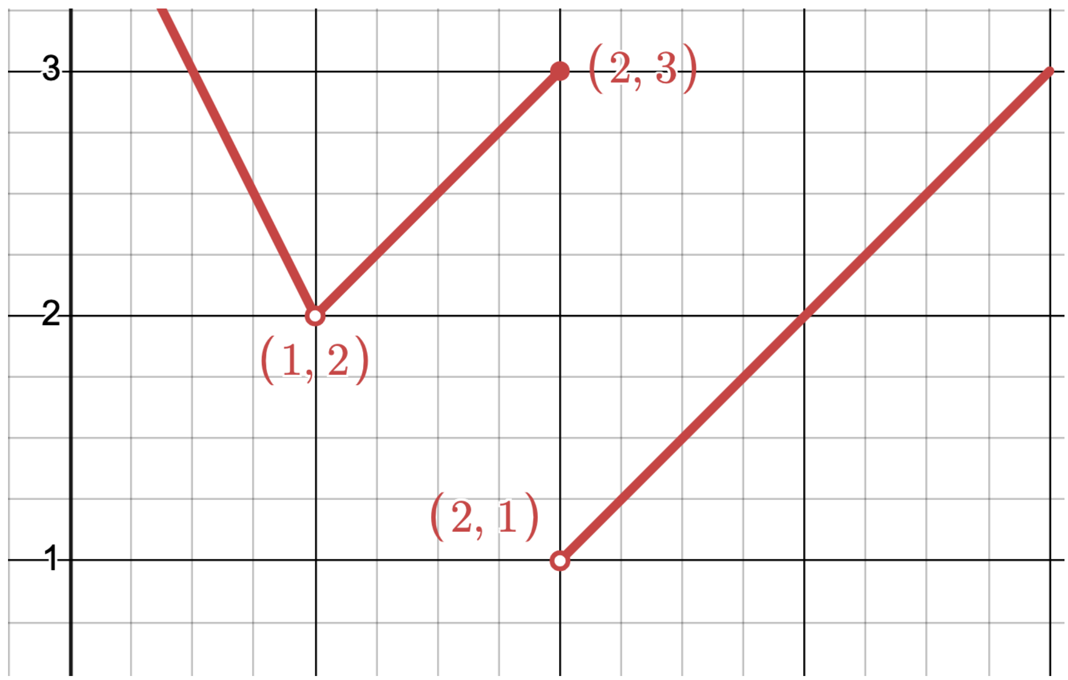


(d) Explain how 5d proves the function $F(x) = x^5 + 1$ has a fixed point between $x = -2$ and $x = -1$.

Approximate the value of the fixed point using desmos rounded to four decimal places.

SECTION 2.7 PRACTICE PROBLEMS

1. For the function below, $\lim_{x\to 1} f(x) = 2$, $\lim_{x\to 2^-} f(x) = 3$, and $\lim_{x\to 2^+} f(x) = 1$.



(a) Estimate the largest $\delta > 0$ so that $|f(x) - 2| < 0.5$ if $0 < |x - 1| < \delta$.

Explain your reasoning using the graph.

(b) Graphically explain why it is impossible to find a $\delta > 0$ so that $|f(x) - 1| < 0.5$ when $0 < |x - 2| < \delta$.

2. Use the precise $(\epsilon - \delta)$ definition of limit to prove $\lim_{x \to 2} \frac{4-5x}{2} = -3$.

SCRATCHWORK:

FORMAL PROOF: